11.5 Pascal's Triangle and the Binomial Theorem

Factorials and the Binomial Coefficient

We define **n factorial** as

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1,$$

where $1! = 1$ and $0! = 1$.

Example. Evaluate $4!$.

Solution. $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. □

We define the **n choose k** as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

This expression is also called the **binomial coefficient.**

Example. Evaluate $\binom{6}{2}$.

Solution.

$$\binom{6}{2} = \frac{6!}{2!(6-2)!} = \frac{6!}{2!4!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(2 \cdot 1)(4 \cdot 3 \cdot 2 \cdot 1)} = \frac{6 \cdot 5}{2} = 3 \cdot 5 = 15.$$ □

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

The main purpose of Pascal's Triangle is to find the coefficients of the expansion of the binomial expression $(a+b)^n$. The first and last number of each row is always 1. If a term is neither the first nor the last, then it can be found by summing the two terms above and diagonal to it.

Binomial Expansion Using Pascal's Triangle

Example. Expand the binomial $(a+b)^3$.

Solution. We can expand using the distributive property.

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)(a+b) = (a+b)(a^2+2ab+b^2)\\ &= a(a^2+2ab+b^2)+b(a^2+2ab+b^2)\\ &= (a^2+2a^2b+ab^2)+a^2b+2ab^2+b^3\\ &= a^3+3a^2b+3ab^2+b^3.\end{aligned}$$ □

That was a lot of work. Notice that the powers of a begin at 3 and descend to 0, while the powers of b begin at 0 and ascend to 3. The coefficients of the expansion are $1, 3, 3, 1$. These are the same numbers that are found on Pascal's Triangle. It turns out that we can use Pascal's Triangle to find the coefficients of a binomial expansion.

Example. Find the binomial expansion of $(a+b)^4$.

Solution. For the coefficients of the expansion, go to the line of Pascal's Triangle that begins with $1, 4, \ldots$ This is the row that we want. Reading off the row, we get the coefficients $1, 4, 6, 4, 1$. The powers of a descend, a^4, a^3, a^2, a, 1. The powers of b ascend, 1, b, b^2, b^3, b^4. Putting everything together gives

$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4.$$ □

Example. Find the binomial expansion of $(a-b)^4$.

Solution. In the previous example, we found the expansion of $(a+b)^4$. We can replace b by $-b$ in that expansion to get the answer.

$$\begin{aligned}(a-b)^4 &= (a+(-b))^4 = a^4 + 4a^3(-b) + 6a^2(-b)^2 + 4a(-b)^3 + (-b)^4 \\ &= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4\end{aligned}$$ □

The Binomial Theorem

The next theorem gives a formula for finding the coefficients of each term of a binomial expansion.

Theorem 1 (The Binomial Theorem)**.** *The expansion of the binomial* $(a+b)^n$ *is given by*

$$\begin{aligned}(a+b)^n &= \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{k}a^{n-k}b^k + \cdots + \binom{n}{n}b^n \\ &= \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k.\end{aligned}$$

Example. Find the coefficient for the term a^3b^5 in the expansion of $(a-2b)^8$.

Solution. We can write

$$(a-2b)^8 = (a+(-2b))^8.$$

When the expression is expanded, the term containing a^3b^5 is given by $\binom{n}{k} = \binom{8}{3}$. The term we want is then

$$\binom{8}{3}a^3(-2b)^5 = \frac{8!}{3!(8-3)!}(-2)^5a^3b^5.$$

The coefficient is then

$$\frac{8!}{3!5!}(-32) = \frac{(8\cdot 7\cdot 6)\cdot 5!}{3!5!}(-32) = \frac{8\cdot 7\cdot 6}{6}(-32) = -56\cdot 32 = -1792.$$ □