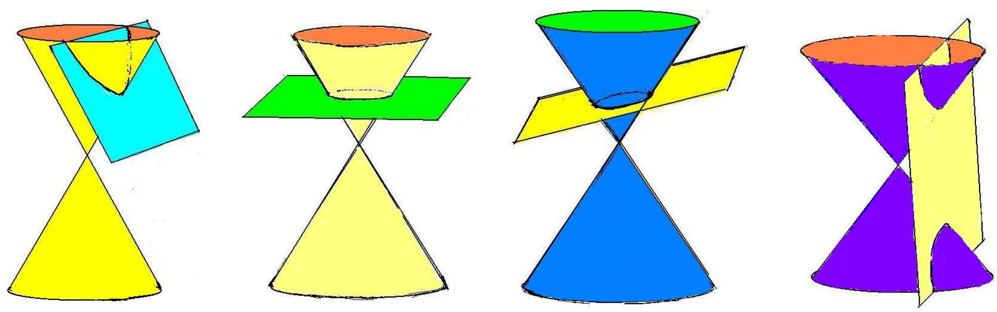
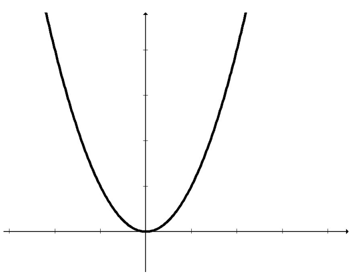
# 10 Conic Sections

A conic section is the intersection of a plane and a double-napped cone. The four basic conics are the circle, ellipse, parabola, and hyperbola.



$y = x^2$



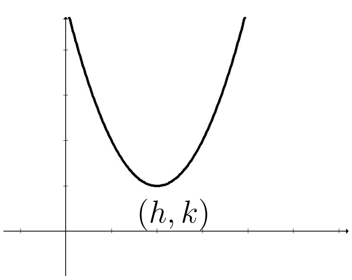
## Parabolas

The basic parabola is $y = x^2$. We can plot points to sketch the graph. The graph was studied in detail in a previous section, so we'll just go over the basics.

The standard form of a parabola is $y = a(x-h)^2 + k$, where the vertex is $(h, k)$. The parabola faces up if $a > 0$, and down if $a < 0$.

The general form of a parabola is $y = ax^2 + bx + c$. By completing the square the equation can be converted to standard form. The first coordinate of the vertex, $h$, can be found using the formula $h = -b/2a$. The second coordinate of the vertex, $k$, can then be found by putting $h$ back in for $x$ in the formula $y = ax^2 + bx + c$.

$y = (x-h)^2 + k$

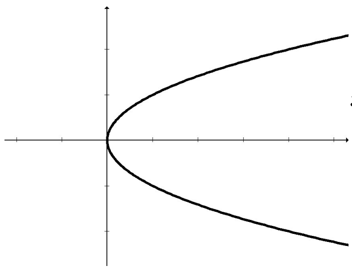



The equation for a parabola can also be written as

$$(y - k) = a(x - h)^2,$$

where the vertex is $(h, k)$.

Let's now look at what happens when we interchange the $x$ and $y$ variables in the equation $y = x^2$. We get the inverse relation $x = y^2$. The graph of the inverse relation is found by reflecting the graph of $y = x^2$ about the line $y = x$. The result




is a parabola with vertex at the origin and that opens up along the positive $x$-axis.

Replacing $x$ by $(x - h)$ results in a horizontal translation, while replacing $y$ by $(y - k)$ results in a vertical translation. The standard form is

$$(x - h) = a(y - k)^2,$$

or

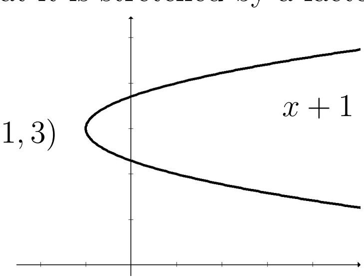
$$x = a(y - k)^2 + h.$$

The vertex is $(h, k)$. If $a > 0$ it opens to the right. If $a < 0$ it opens to the left.

**Example.** Sketch the graph of the parabola. Label the vertex.

$$x + 1 = 2(y - 3)^2$$

*Solution.* Here we have $h = -1$, $k = 3$, and $a = 2$. The vertex is $(-1, 3)$ and it opens to the right. Plotting points will show that it is stretched by a factor of $a = 2$.

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The general form of a parabola opening horizontally is

$$x = ay^2 + by + c$$

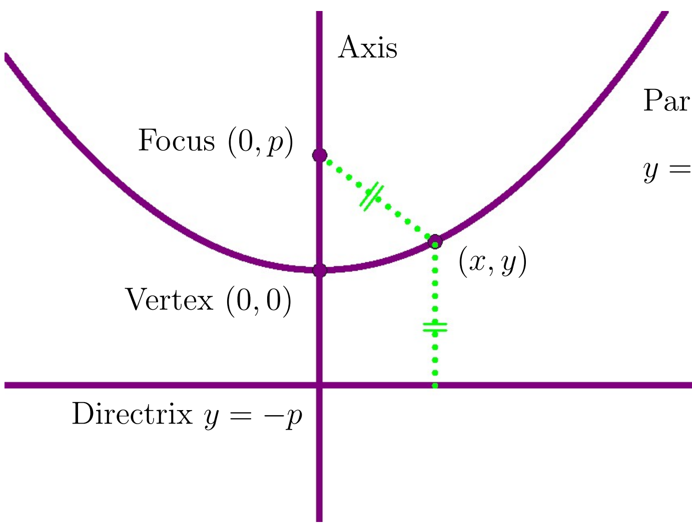
To write this in this in standard form, we can complete the square as was shown in the section on quadratics. We can also use the formula $k = -b/2a$, and then solving for $h$ by putting

$y = k$ into the equation. (Recall that for $y = ax^2 + bx + c$, we have $h = -b/2a$).

The geometric interpretation of a parabola is interesting and has applications to physics and engineering.

**Definition 1.** *A parabola is the set of all points $(x, y)$ in the plane that are equidistant from a fixed line, called the directrix, and a fixed point, called the focus.*

The vertex is the midpoint between the focus and the directrix. The axis of the parabola is the line passing through the focus and the vertex.




Parabola with vertex at the origin.

$y = \frac{1}{4p}x^2$

If the focus is located at $(0, p)$ and the directrix is given by the equation $y = -p$, then the parable is the set of all points $(x, y)$ equidistant from the points $(0, p)$ and the line $y = -p$. The distance between the points $(x, y)$ and $(0, p)$ is

$$\sqrt{(x-0)^2 + (y-p)^2},$$

while the distance from the point $(x, y)$ and the line $y = -p$ is

$$(y - (-p)) = y + p.$$

The point $(x, y)$ lies on the parabola if and only if these two distances are equal,

$$\sqrt{x^2 + (y-p)^2} = y + p.$$

We can square both sides and simplify.

$$x^2 + (y-p)^2 = (y+p)^2,$$
$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2,$$
$$x^2 - 2py = 2py,$$
$$x^2 = 4py,$$
$$y = \frac{1}{4p}x^2.$$

The standard equation of a parabola with vertex $(0, 0)$ and directrix
$y = -p$ is

$$y = \frac{1}{4p}x^2, \ p \neq 0.$$

If we let $a = \frac{1}{4p}$, then we get the equation $y = ax^2$. If $a > 1$, then the graph is stretched by a factor of $a$. If $0 < a < 1$, then the graph is compressed by a factor of $a$. If $a < 0$, then the graph is reflected about the $x$ axis.

## Circles

**Definition 2.** *A circle is the set of all points $(x, y)$ in the plane with constant distance $r$ from a point, $(h, k)$, the center.*

An equation of a curve is an equation satisfied by the coordinates of the points on the curve and by no other points. Let's use the distance formula to find the equation of a circle with radius $r$ and center $(h, k)$. By definition, the circle is the set of all points $P(x, y)$ whose distance from the center $C(h, k)$ is

$r$. Thus, $P$ is on the circle if and only if $|PC| = r$. From the distance formula, we have

$$\sqrt{(x-h)^2 + (y-k)^2} = r$$

or equivalently, squaring both sides, we get

$$(x-h)^2 + (y-k)^2 = r^2.$$

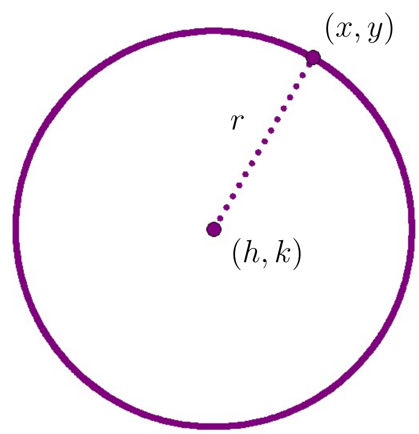
This is the desired equation.

An equation of the circle with center $(h, k)$ and radius $r$ is

$$(x-h)^2 + (y-k)^2 = r^2.$$

In particular, if the center is the origin $(0, 0)$, the equation is

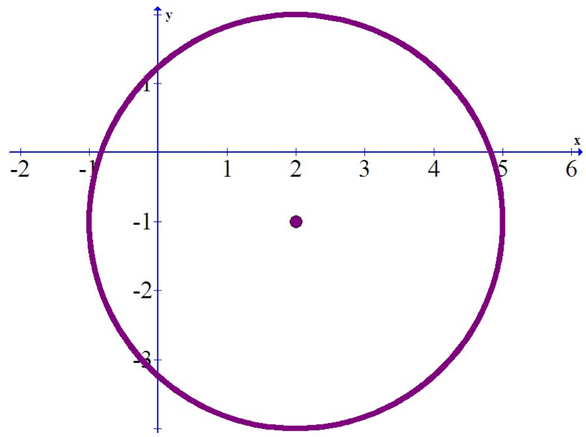
$$x^2 + y^2 = r^2.$$




**Example.** Write the standard form of the equation of a circle with center $(2, -1)$ and radius $r = 3$. Sketch the circle.

*Solution.*

$$(x-2)^2 + (y+1)^2 = 3^2$$

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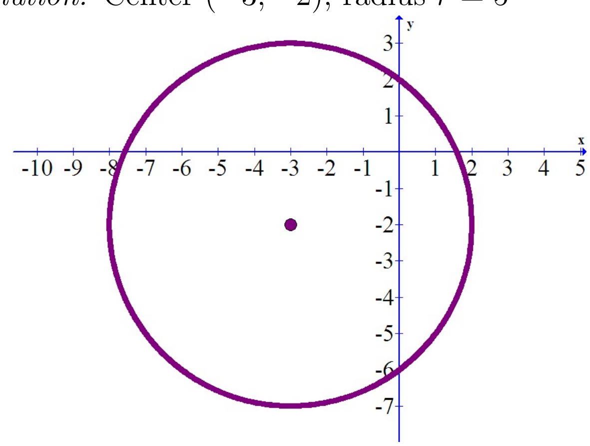
**Example.** Find the center and radius of the circle. Sketch the graph.

$$(x+3)^2 + (y+2)^2 = 25$$

*Solution.* Center $(-3, -2)$, radius $r = 5$

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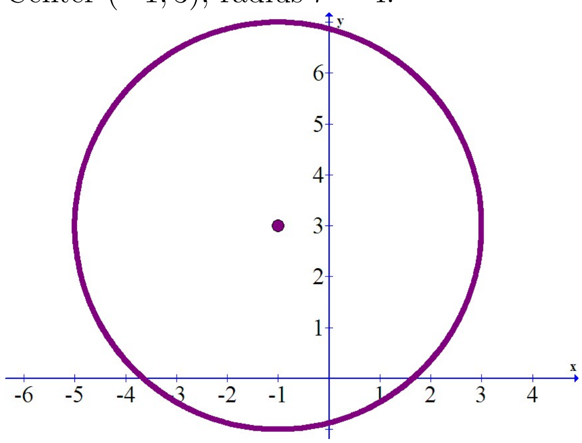
**Example.** Write the equation for the circle in standard form. State the center and the radius. Sketch the graph.

$$x^2 + 2x + y^2 - 6y = 6$$

*Solution.* We will complete the square for both the $x$ terms and the $y$ terms.

$$\begin{aligned} x^2+2x+y^2-6y &= 6 \\ (x^2+2x+1)+(y^2-6y+9) &= 6+1+9 \\ (x+1)^2+(y-3)^2 &= 16 \end{aligned}$$

Center $(-1,3)$, radius $r=4$.

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## Ellipses

An ellipse looks like an elongated circle. The equation of a circle centered at the origin with radius $r=1$ is

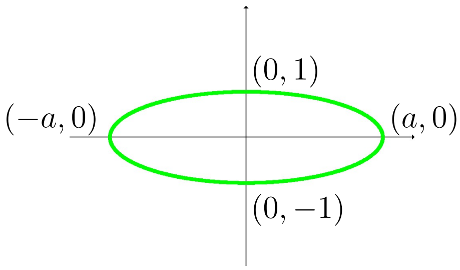
$$x^2+y^2=1.$$

If $a$ is a positive, then replacing $x$ by the expression $x/a$ will stretch the graph of the circle in the horizontal direction,

$$\left(\frac{x}{a}\right)^2+y^2=1,$$

$$\frac{x^2}{a^2}+y^2=1.$$

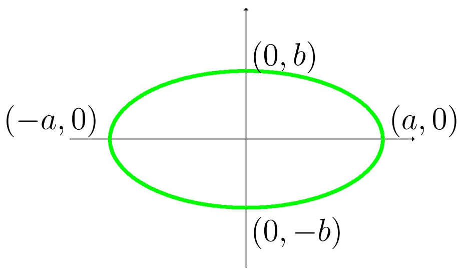
Let's plot some points to find the $x$- and $y$-intercepts and then sketch the graph. When $y=0$, we get $x^2/a^2=1$, so that $x^2=a^2$, and $x=\pm 1$.




When $x = 0$, we get $y^2 = 1$, so that $y = \pm 1$. The graph is shown to the left.

Let $b$ be a positive constant. If we replace $y$ by $y/b$, then the graph will be stretched in the vertical direction,

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1.$$




The **standard form of an ellipse centered at the origin** is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{1}$$

Again, we plot some points to find the $x$- and $y$-intercepts and then sketch the graph. When $y = 0$, we get $x^2/a^2 = 1$, so that $x^2 = a^2$, and $x = \pm 1$. When $x = 0$, we get $y^2/b^2 = 1$, so that $y = \pm b$. The graph is shown to the left.

**Example.** Sketch the ellipse given by

$$4x^2 + y^2 = 36,$$

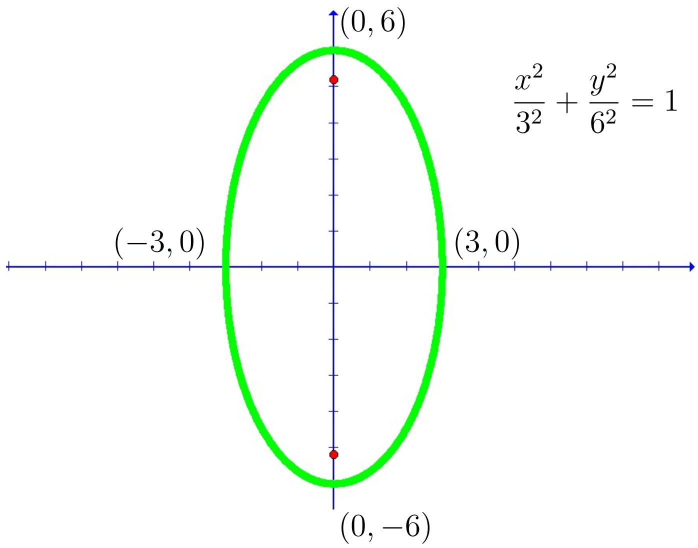
and find the $x$- and $y$-intercepts.

*Solution.* First, rewrite as $\dfrac{x^2}{3^2} + \dfrac{y^2}{6^2} = 1$. We see that $b = 3$ and $a = 6$.

One quick way to make a quick sketch of an ellipse centered at the origin is to find the $x$- and $y$-intercepts and then sketch the ellipse using the intercepts as a guide.

| | |
|---|---|
| $x$-intercepts: set $y = 0$. | $y$-intercepts: set $x = 0$. |
| $4x^2 + y^2 = 36$ | $4x^2 + y^2 = 36$ |
| $4x^2 + 0^2 = 36$ | $4(0)^2 + y^2 = 36$ |
| $x^2 = 9$ | $y^2 = 36$ |
| $x = \pm 3$ | $y = \pm 6$ |
| The $x$-intercepts are $(\pm 3, 0)$. | The $y$-intercepts are $(0, \pm 6)$. |

We have $a = 3$ and $b = 6$

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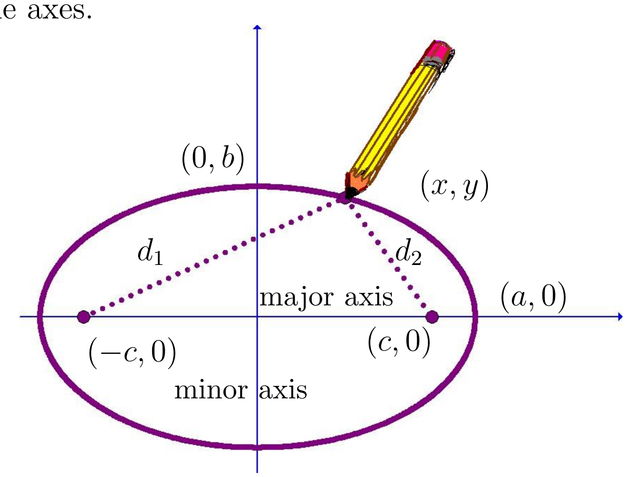
A very nice way to define an ellipse is by a geometric interpretation. This definition is very useful in applications like astronomy.

**Definition 3.** *An ellipse is the set of all points $(x, y)$ in a plane the sum of whose distances from two distinct points (foci) is constant.*

Hammer two nails into a board. The nails will be the foci. Tie the ends of a string to each of the nails. Then take a pencil and stretch the string until it is tight. Start drawing, and you get an ellipse.

The axes of an ellipse are the horizontal and vertical line segments of maximum length whose endpoints are on the ellipse. The axis of greater length is called the major axis. The axis of lesser length is called the minor axis. The major axis will pass through the foci. The endpoints of the major axis are called the vertex. The center of an ellipse is the point of intersection of the axes.




Center $(0, 0)$

Foci $(-c, 0)$, $(c, 0)$

We will now develop formula (1)for the equation of an ellipse centered at the origin by using Definition 3.

Let $d_1$ be the distance from a point $(x, y)$ on the ellipse to the foci $(-c, 0)$, and let $d_2$ be the distance from the same point $(x, y)$ to the foci $(c, 0)$. We can calculate $d_1$ and $d_2$ by using the distance formula,

$$d_1 = \sqrt{(x - (-c))^2 + (y - 0)^2},$$
$$d_2 = \sqrt{(x - c)^2 + (y - 0)^2}.$$

A point $(x, y)$ lies on the ellipse if and only if the sum $d_1 + d_2$ is constant. Let's label that constant as $2a$, so that $d_1 + d_2 = 2a$. We then have

$$d_1 + d_2 = a$$
$$\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a$$

First let's move one of the radical expressions to the right hand side,

$$\sqrt{(x + c)^2 + y^2} = 2a - \sqrt{(x - c)^2 + y^2}.$$

If we square both sides, we get

$$(\sqrt{(x+c)^2+y^2})^2 = (2a - \sqrt{(x-c)^2+y^2})^2,$$
$$(x+c)^2+y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + (x-c)^2+y^2,$$
$$x^2+2cx+c^2+y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + x^2 - 2cx + c^2 + y^2,$$
$$a^2 - cx = a\sqrt{(x-c)^2+y^2},$$

We then square both sides again.

$$(a^2-cx)^2 = a^2((x-c)^2+y^2),$$
$$a^4 - 2a^2cx + c^2x^2 = a^2(x^2-2cx+c^2+y^2),$$
$$a^4 - 2a^2cx + c^2x^2 = a^2x^2 - 2a^2cx + a^2c^2 + a^2y^2,$$
$$a^4 + c^2x^2 = a^2x^2 + a^2c^2 + a^2y^2,$$
$$(a^2-c^2)x^2 + a^2y^2 = a^2(a^2-c^2).$$

If we let $b^2 - a^2 - c^2$, then we get

$$b^2x^2 + a^2y^2 = a^2b^2.$$

Dividing through by $a^2b^2$ gives the standard form of an ellipse centered at the origin.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The *standard equation of an ellipse centered at the origin with major and minor axes of lengths* $2a$ *and* $2b$, $0 < b < a$ *is*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The vertices and foci lie on the major axis, $a$ and $c$ units, respectively, form the center. The numbers $a, b$, and $c$ satisfy

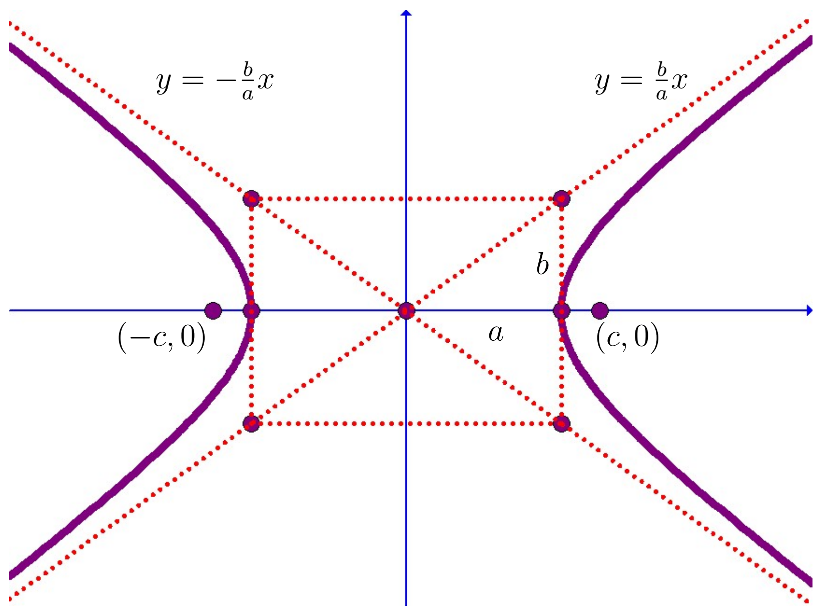
$$c^2 = a^2 - b^2$$

The constant $a$ is always greater than the constant $b$. If $a$, the greater number, is in the denominator of the $x^2$ term, then the ellipse will be elongated along the $x$-axis. On the other hand, if $a$, the greater number, is in the denominator of the $y^2$ term, then the ellipse will be elongated along the $y$-axis.

## Hyperbolas

**Definition 4.** *A hyperbola is the set of all points $(x, y)$ in plane the difference of whose distances form two distinct points (foci) is a positive constant.*

The graph has two disconnected parts (branches). The line through the two foci intersect the hyperbola at two points (vertices). The line segment connecting the vertices is the transverse axis, and the midpoint of the transverse axis is the center of the hyperbola.




The standard equation of a hyperbola (centered at the origin) is given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{Transverse axis horizontal}$$

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \quad \text{Transverse axis vertical}$$

The vertices and foci are, respectively, $a$ and $c$ units form the center. Moreover, $a$, $b$, and $c$ are related by $a^2 + b^2 = c^2$.

If we solve for $y$ in the case where the transverse axis is horizontal, we get

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
$$\frac{x^2}{a^2} - 1 = \frac{y^2}{b^2}$$
$$y^2 = \frac{b^2x^2}{a^2} - b^2$$
$$y^2 = \frac{b^2x^2}{a^2}\left(1 - \frac{a^2}{x^2}\right)$$
$$y = \pm\frac{b}{a}x\sqrt{1 - \frac{a^2}{x^2}}$$

As $x$ gets very large, $a^2/x^2$ gets very close to zero. This gives asymptotes

$$y = \pm\frac{b}{a}x$$

The easiest way to find the equation for the asymptotes to the hyperbola is to replace 1 by 0 in the equations

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{or} \quad \frac{y^2}{b^2} - \frac{x^2}{a^2} = 1.$$

This gives

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \quad \text{or} \quad \frac{y^2}{b^2} - \frac{x^2}{a^2} = 0.$$

Both of these gives $y = \pm\frac{b}{a}x$.

We can also sketch the asymptotes by drawing a rectangle with vertices $(\pm a, \pm b)$. The asymptotes will pass through the vertices of the rectangle.

**Example.** Write the hyperbola in standard form. Sketch the hyperbola, labelling intercepts and drawing asymptotes.

$$4x^2 - y^2 = 16$$

*Solution.* Dividing through by 16 gives

$$\frac{x^2}{4} - \frac{y^2}{16} = 1$$

$$\frac{x^2}{2^2} - \frac{y^2}{4^2} = 1$$

We have that $a = 2$ and $b = 4$.

We can see that the hyperbola opens along the $x$-axis by setting $y = 0$. We get

$$4x^2 = 16$$

$$x^2 = 4$$

$$x = \pm 2$$

Therefore the $x$ intercepts are $(-2, 0)$ and $(2, 0)$. The center is the origin.

The foci are found by using $a^2 + b^2 = c^2$. We get

$$c^2 = 2^2 + 4^2$$

$$c = \sqrt{20} = 2\sqrt{5}$$

The foci are $(\pm 2\sqrt{5}, 0)$.

The asymptote is found by replacing 1 by 0 in the equation of the hyperbola.

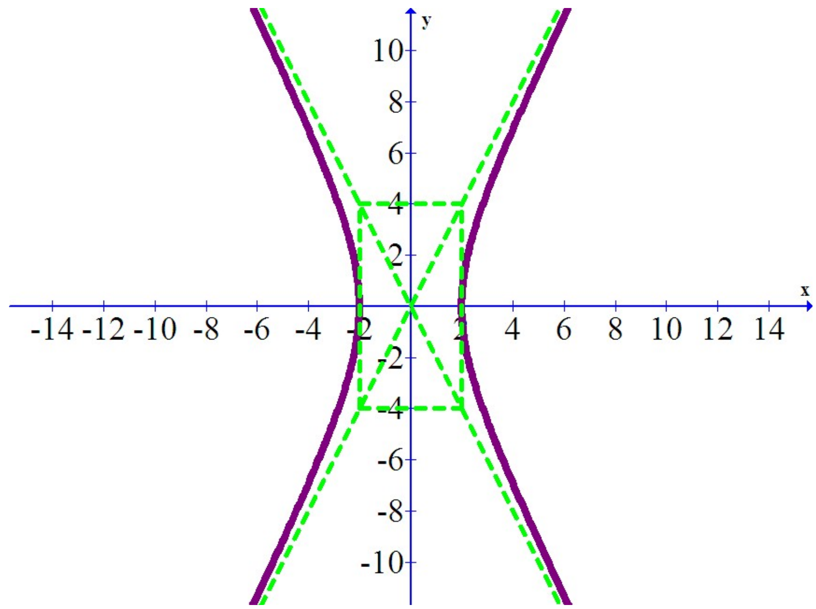
$$\frac{x^2}{2^2} - \frac{y^2}{4^2} = 0$$

$$\frac{x^2}{2^2} = \frac{y^2}{4^2}$$

$$y = \pm 2x$$

We can sketch the graph by plotting the $x$ intercepts, $(-2, 0)$ and $(2, 0)$. We can also find the asymptotes by drawing a box with vertices $(\pm 2, \pm 4)$. The asymptotes will pass the the vertices of the rectangle. By looking at the rectangle that we have drawn, we can see that the slope of the asymptotes are $m = \pm 2$

and that the $y$-intercept of the asymptotes are $b = 0$. Therefore the equations of the asymptotes are $y = \pm 2x$.

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