4.4 Exponential and Logarithmic Equations

Find all of the real solutions.

1. $\log(3x+1) = 2$

   SOLUTION

$$\begin{aligned} 10^2 &= 3x+1 \\ 3x+1 &= 100 \\ 3x &= 99 \\ x &= 33 \end{aligned}$$

2. $\log(x+1) - \log(x) = 3$

   SOLUTION

$$\begin{aligned} \log\frac{x+1}{x} &= 3 \\ \frac{x+1}{x} &= 10^3 \\ x+1 &= 100x \\ 99x &= 1 \\ x &= 1/99 \end{aligned}$$

3. $\log_2(x+2) + \log_2(x-2) = 5$

   SOLUTION

$$\begin{aligned} \log_2(x+2)(x-2) &= 5 \\ (x+2)(x-2) &= 2^5 \\ x^2-4 &= 32 \\ x^2 &= 36 \\ x &= \pm 6 \\ x &= 6 \quad \text{Note that } \log_2(-6+2) \text{ is undefined} \end{aligned}$$

4. $\ln(x) + \ln(x+2) = \ln 8$

SOLUTION

$$\begin{aligned}
\ln x(x+2) &= \ln 8 \\
x(x+2) &= 8 \\
x^2 + 2x &= 8 \\
x^2 + 2x - 8 &= 0 \\
(x+4)(x-2) &= 0 \\
x = -4 & \quad x = 2 \\
x &= 2 \quad \text{Note that } \ln(-2) \text{ is undefined}
\end{aligned}$$

5. $\ln x - \ln(x+1) = \ln(x+3) - \ln(x+5)$

SOLUTION

$$\begin{aligned}
\ln x/(x+1) &= \ln(x+3)/(x+5) \\
x/(x+1) &= (x+3)/(x+5) \\
x(x+5) &= (x+3)(x+1) \\
x^2 + 5x &= x^2 + 4x + 3 \\
x &= 3
\end{aligned}$$

6. $2^{x-1} = 7$

SOLUTION

$$\begin{aligned}
\ln 2^{x-1} &= \ln 7 \\
(x-1)\ln 2 &= \ln 7 \\
x\ln 2 - \ln 2 &= \ln 7 \\
x \ln 2 &= \ln 7 + \ln 2 \\
x &= \frac{\ln 7 + \ln 2}{\ln 2}
\end{aligned}$$

7. $9 = e^{-3x}$

SOLUTION

$$\begin{aligned} \ln 9 &= \ln e^{-3x} \\ \ln 9 &= -3x \\ x &= -\frac{1}{3} \cdot \ln 9 \end{aligned}$$

8. $7 = 2e^{3t}$

SOLUTION

$$\begin{aligned} 7/2 &= e^{3t} \\ \ln(7/2) &= 3t \\ x &= \frac{1}{3} \cdot \ln(7/2) \end{aligned}$$

9. $5 = 7e^{2t}$

SOLUTION

$$\begin{aligned} 5/7 &= e^{2t} \\ \ln(5/7) &= 2t \\ x &= \frac{1}{2} \cdot \ln(5/7) \end{aligned}$$

10. $2^x = 3^{x-1}$

SOLUTION

$$\begin{aligned} \ln(2^x) &= \ln 3^{x-1} \\ x \ln 2 &= (x-1) \ln 3 \\ x \ln 2 &= x \ln 3 - \ln 3 \end{aligned}$$

$$\begin{aligned} x\ln 2 - x\ln 3 &= -\ln 3 \\ x(\ln 2 - \ln 3) &= -\ln 3 \\ x &= \frac{-\ln 3}{\ln 2 - \ln 3} \end{aligned}$$

11. Exponential Decay is given by the function

$$A(t) = A_0 e^{-kt}$$

where A is the amount at time t, t is time in years, A_0 is the amount at time $t = 0$, and k is a constant that depends only on the material that is decaying exponentially.

The half-life, h, of a material is the time that it takes for half of a given amount of material to decay. Find a formula that gives the constant k in terms of the half-life h.

SOLUTION

Let $A_0 = 1$, $A = 1/2$, $t = h$. Find k.

$$1/2 = 1 \cdot e^{-kh}$$

$$\ln 1/2 = -kh$$
$$\ln 2^{-1} = -kh$$
$$-1 \cdot \ln 2 = -kh$$
$$k = \frac{\ln 2}{h}$$

12. Carbon-14 decays exponentially with a half-life of $h = 5730$ years. Write the constant k in terms of the half-life. How long does it take 2.4 g of carbo-14 to be reduced to 1.3 g of carbon-14?

SOLUTION $k = \frac{\ln 2}{h} = \frac{\ln 2}{5730}$

$A_0 = 2.4$, $A = 1.3$, $k = (\ln 2)/5730$, find t.

$$\begin{aligned}
A(t) &= A_0 e^{-kt} \\
1.3 &= 2.4 e^{-\frac{\ln 2}{5730}t} \\
\frac{1.3}{2.4} &= e^{-\frac{\ln 2}{5730}t} \\
\ln \frac{1.3}{2.4} &= -\frac{\ln 2}{5730}t \\
t &= -\left(\frac{5730}{\ln 2}\right)\left(\ln \frac{1.3}{2.4}\right) \\
t &\approx 5068.3 \text{ years}
\end{aligned}$$

13. *Dating a Bone:* A piece of bone from an organism is found to contain 10% of the carbon-14 that it contained when it was living. How long ago was the organism alive?

    Solution $k = \frac{\ln 2}{h} = \frac{\ln 2}{5730}$

    $A_0 = 1$, $A = .1$, $k = (\ln 2)/5730$, find t.

$$\begin{aligned}
A(t) &= A_0 e^{-kt} \\
.1 &= 1 \cdot e^{-\frac{\ln 2}{5730}t} \\
\ln .1 &= -\frac{\ln 2}{5730}t \\
t &= -\left(\frac{5730}{\ln 2}\right)(\ln .1) \\
t &\approx 19,035 \text{ years}
\end{aligned}$$

14. *Radioactive Waste:* If 25 g of radioactive waste reduces to 20 g in 8000 years, then what is the half-life, h, for the radioactive element?

    Solution

$$A(t) = A_0 e^{-kt}$$

$$
\begin{aligned}
A = 25, & \quad A = 20, \quad t = 8000 \\
20 &= 25e^{-k\cdot 8000} \\
20/25 &= e^{-k\cdot 8000} \\
4/5 &= e^{-k\cdot 8000} \\
\ln(.8) &= -k \cdot 8000 \\
k &= -\frac{\ln(.8)}{8000} \\
k &= \frac{\ln 2}{h} \\
-\frac{\ln(.8)}{8000} &= \frac{\ln 2}{h} \\
h &= -\frac{8000 \ln 2}{\ln(.8)} \\
h &\approx 24,850 \text{ years}
\end{aligned}
$$

15. Interest compounded continuously is given by the formula

$$A = Pe^{rt}.$$

A principal balance P is invested at a constant yearly rate r compounded continuously. If it takes 5 years for the principal to double, then what is the yearly interest rate r?

SOLUTION

$$
\begin{aligned}
A &= Pe^{rt} \\
A = 2 & \quad P = 1, \quad t = 5 \\
2 &= 1 \cdot e^{5r} \\
\ln 2 &= 5r \\
r &= \frac{\ln 2}{5} \\
r &\approx .13863 = 13.863\%
\end{aligned}
$$

16. The population P of a city grows exponentially according to the function

$$P(t) = P_0 e^{kt}$$

where $P_0 = 30,000$ is the population at the starting time of January 1, 2000, t is the number of years after that starting time, and k is a constant. Find the constant k if it takes 11 years for the population to double. What will be the population at the beginning of 2017?

SOLUTION

$$P = 60,000 \qquad P_0 = 30,000, \quad t = 11 \qquad \text{find } k$$

$$\begin{aligned} P(t) &= P_0 e^{kt} \\ 60,000 &= 30,000 \cdot e^{11k} \\ 2 &= e^{11k} \\ \ln 2 &= 11k \\ k &= \frac{\ln 2}{11} \end{aligned}$$

A general formula is $k = \frac{\ln 2}{d}$ where d is the doubling time.

$$\begin{aligned} P_0 &= 30,000, \quad t = 17, \quad k = \frac{\ln 2}{11}, \quad \text{find } P \\ P &= 30,000 \cdot e^{\frac{\ln 2}{11} \cdot 17} \\ P &= 87,569 \text{ people} \end{aligned}$$