

## 4.1 Exponential Functions

### The Graph of Exponential Functions

In this section we study functions of the form  $y = a^x$ , where  $a$  is a positive number. Let's begin by plotting points and then sketching the graph of the function exponential function  $y = 2^x$ . We make a table and plot points.

$x$	$y = 2^x$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$
4	$2^4 = 16$

If we start at the bottom and move to the top, we have the sequence

$$16, 8, 4, 2,$$

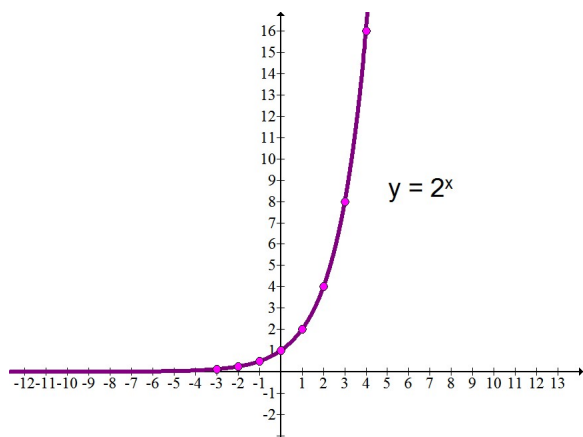
If we continue with this sequence, we get

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$$

$x$	$y = 2^x$
-4	$2^{-4} = 1/2^4 = 1/16$
-3	$2^{-3} = 1/2^3 = 1/8$
-2	$2^{-2} = 1/2^2 = 1/4$
-1	$2^{-1} = 1/2$
0	$2^0 = 1$

So far we have only plotted integers. If  $x$  is a rational function, that is, if  $x$  is of the form  $m/n$ , where  $m$  and  $n$  are integers, then we have  $2^{m/n} = \sqrt[n]{2^m}$ . For values of  $x$  that cannot be expressed as a fraction of integers, that is, for irrational values of  $x$ , we fill in the graph using nearby points.

Here is the graph of  $y = 2^x$  drawn using a computer program.



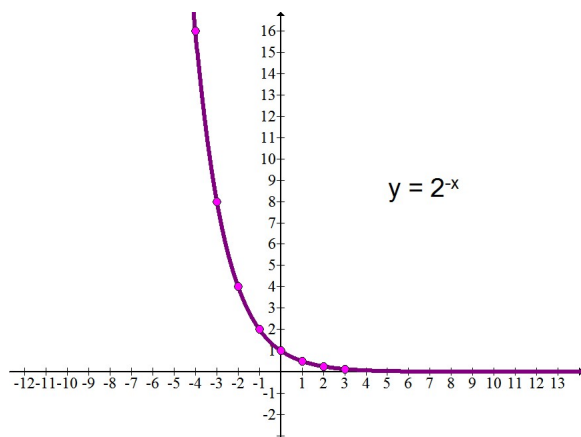
We see that as  $x$ -values increases by 1, the  $y$ -values double. We can make a list of some of the properties of the function  $y = 2^x$ . An exponential function of the form  $y = a^x$ , where  $a > 1$  will have these same properties.

- Domain: All real numbers.
- Range: All  $y > 0$ .
- Increasing
- One-to-one
- Invertible
- Horizontal Asymptote: The  $x$ -axis.

Next, let's plot points and sketch the exponential function  $y = 2^{-x}$ .

$x$	$y = 2^{-x}$
-4	16
-3	8
-2	4
-1	2
0	$2^0 = 1$

$x$	$y = 2^{-x}$
1	$2^{-1} = 1/2$
2	$2^{-2} = 1/2^2 = 1/4$
3	$2^{-3} = 1/2^3 = 1/8$
4	$2^{-4} = 1/2^4 = 1/16$
5	$2^{-5} = 1/2^5 = 1/32$

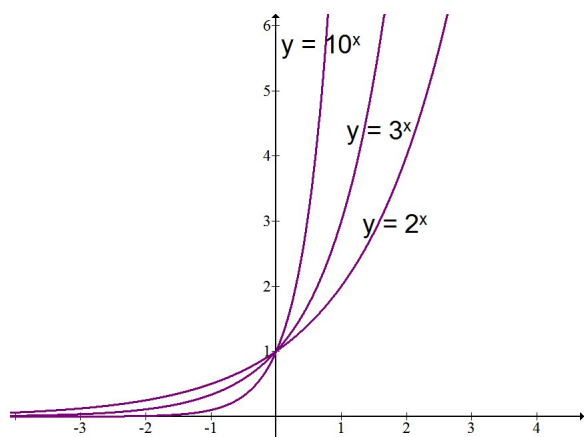


Note that by replacing  $x$  by  $-x$  in the function  $y = 2^x$ , the graph is reflected about the  $y$ -axis. We see that as  $x$ -values decrease by 1,  $y$ -values are halved. Also, note that  $y = 2^{-x} = \left(\frac{1}{2}\right)^x$ .

The exponential  $y = 2^x$  is a nice example of an exponential function because it doubles every time the input increases by 1. The number 2 is called the base of  $y = 2^x$ . However, we can use other bases for exponential functions. We should always use a positive number for the base because otherwise there is the problem of raising a negative number to the  $1/2$  power because the result is not a real number. For example  $(-2)^{1/2} = \sqrt{-2}$  which is not a real number.

If the base is a positive number less than 1, say  $1/2$ , then we have  $y = \left(\frac{1}{2}\right)^x = 2^{-x}$ . Because of this, we can always choose the base to be a number greater than 1.

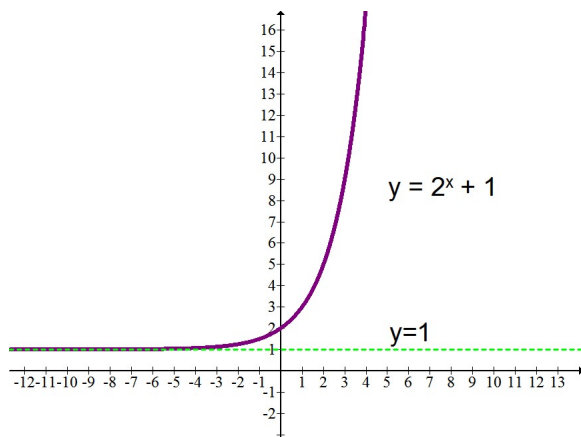
Here are the graphs of three exponential functions,  $y = 2^x$ ,  $y = 3^x$ , and  $y = 10^x$  all sketched on the same plane. Note that they all have the same  $y$ -intercept. This is because  $2^0 = 3^0 = 10^0 = 1$ . Any nonzero number raised to the zero power equals 1.



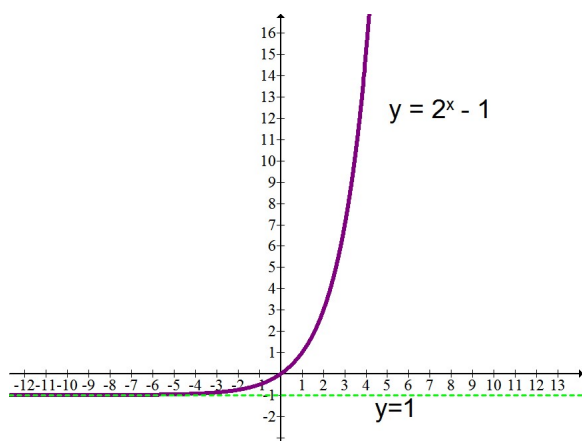
### Transformations of Exponential Functions

**Example** Sketch the graph of the exponential function.

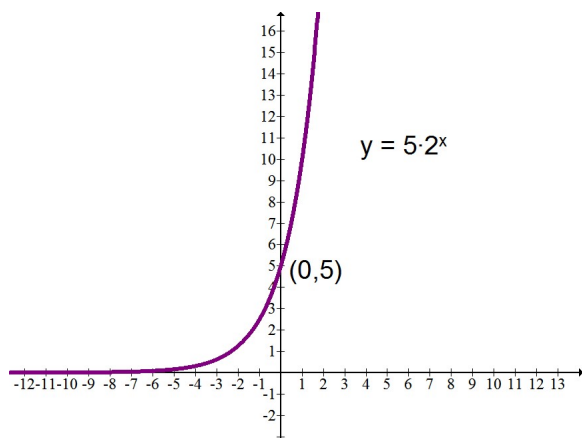
1.  $y = 2^x + 1$



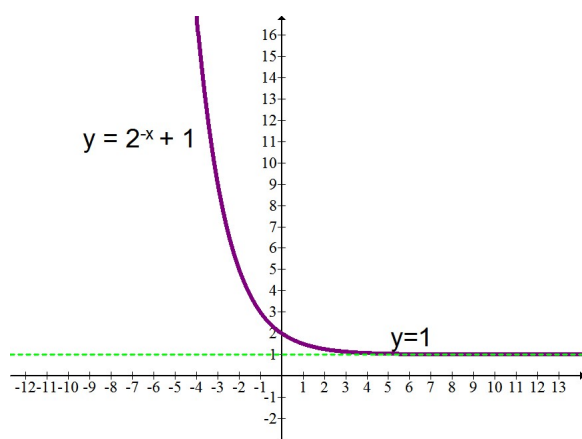
2.  $y = 2^x - 1$



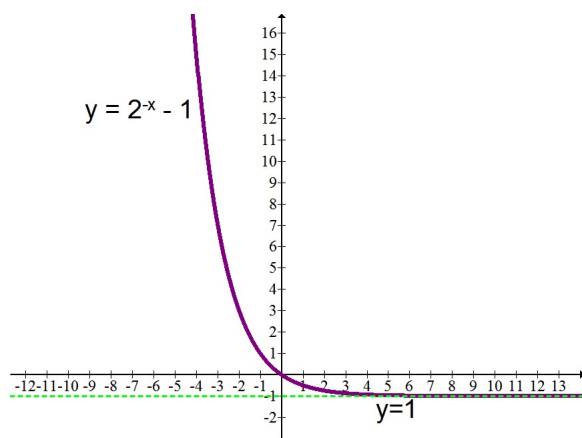
3.  $y = 5 \cdot 2^x$



4.  $y = 2^{-x} + 1$



5.  $y = 2^{-x} - 1$



### Algebra with Exponentials

First let's make a list of some properties of exponents.

**Properties of Exponents** If  $a$  and  $b$  are positive real numbers and  $x$  and  $y$  are also real numbers, then

- $a^x a^y = a^{x+y}$
- $\frac{a^x}{a^y} = a^{x-y}$

- $(a^x)^y = a^{xy}$
- $(ab)^x = a^x a^y$
- $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$
- $a^{-1} = \frac{1}{a}$

We can use these properties to solve the following equations that involve exponents.

**Example** Solve each equation.

1.  $3^x = 27$

SOLUTION  $3^x = 3^3$ .  $x = 3$ .

2.  $10^x = 0.01$

SOLUTION  $10^x = 10^{-2}$ ,  $x = -2$

3.  $2^x = 1/16$

SOLUTION  $2^x = 2^{-4}$ ,  $x = -4$ .

### Compound Interest

**Example** Suppose that \$1000 is invested in a saving account and at the end of each year 3% interest is compounded and added to the balance. That is, \$1000 is invested at an annual interest rate of  $r = 3\%$  compounded annually. What will be the balance at the end of the 5 years?

SOLUTION

At time  $t = 0$ , the balance is \$1000. Then at the end of 1 year, 3% interest is compounded and added to the balance. The new balance is  $\$1000 + .03 \cdot \$1000 = 1000 \cdot (1.03) = 1030$ . We see that the starting balance

of \$1000 is multiplied by 1.03 to get the new balance.

At the end of the second year, the balance is multiplied by 1.03 to get the new balance. Therefore, the new balance is  $1000(1.03)^2$ . The process is continued. At the end of five years, the balance is  $1000(1.03)^5 = \$1,159.30$

We see that a formula for the amount,  $A$ , after  $t$  years is

$$A(t) = 1000 \cdot (1 + .03)^t$$

**Example** Suppose that \$1000 is invested and the annual interest rate is  $r = 3\%$ , but this time the interest is compounded monthly. Find a formula for the amount after  $t$  years.

**SOLUTION** There are 12 months in a year, so that each time the interest is compounded, the compounded rate should not be the full 3%, but 1/12 of 3%, that is  $.03/12$ . We saw that when the interest is compounded, then the balance is multiplied by 1 plus the rate. Therefore, the balance is multiplied by  $(1 + .03/12)$ . After  $t$  years, the interest is compounded  $12t$  times, so that the balance is given by

$$A(t) = 1000 \cdot (1 + .03/12)^{12t}$$

**Formula** If principal  $P$  dollars is invested at an annual interest rate of  $r$  compounded  $n$  times each year, then the amount  $A$  after  $t$  years is given by

$$A(t) = P \left(1 + \frac{r}{n}\right)^{nt}$$

**Example** Suppose 1000 is invested with an annual interest rate of  $r = 5\%$  compounded daily. What is the balance after 7 years?

**SOLUTION**  $P = 1000$ ,  $r = .05$ ,  $n = 365$ ,  $t = 7$ .

$$A(t) = P \left(1 + \frac{r}{n}\right)^{nt} = 1000 (1 + .05/365)^{365 \cdot 7} = \$1,419.00$$

### Interest Compounded Continuously

In some cases we might want the interest to be compounded every second, or even better, every 1/100th of a second. Let's see what happens when the number of times  $n$  that the interest is compounded in a year becomes very big. By the way, how big is very big? We can manipulate the formula for compound interest.

$$A(t) = P \left( 1 + \frac{r}{n} \right)^{nt} = P \left( 1 + \frac{1}{n/r} \right)^{(n/r) \cdot rt}$$

If we just look at the expression

$$\left( 1 + \frac{1}{n/r} \right)^{n/r}$$

and we let  $x = n/r$ , we get

$$\left( 1 + \frac{1}{x} \right)^x$$

As  $n$  gets really big,  $x$  gets really big. We can make a table of values for  $x$ . The output values are approximate values rounded to the fourth decimal place.

$x$	$\left( 1 + \frac{1}{x} \right)^x$
1	2
10	2.5937
100	2.7048
1000	2.7169
$10^6$	2.7183
$10^9$	2.7183

As  $x$  becomes very large, the expression  $\left( 1 + \frac{1}{x} \right)^x$  approaches the irrational number  $e$ , which, rounded to fourth decimal place is approximately

$$e \approx 2.7183$$

Therefore, for very large  $n$ , we have the balance after  $t$  years

$$A(t) = P \left(1 + \frac{r}{n}\right)^{nt} \text{ approaches } Pe^{rt}.$$

We say that interest is **compounded continuously** if a principal balance  $P$  invested at an annual interest rate  $r$  has ending balance amount  $A$  given by

$$A(t) = Pe^{rt},$$

where  $e \approx 2.7183$ .

**Example** A principal amount of \$2000 is invested at a rate of 9% compounded continuously for a 40 years. What is the ending amount?

**SOLUTION**  $P = 2000$ ,  $r = .09$ ,  $t = 40$ ,

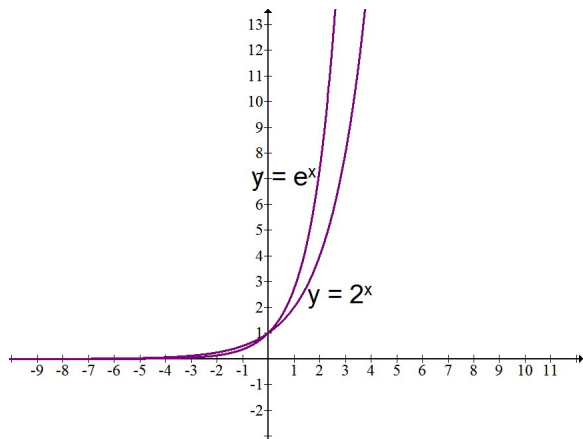
$$A = Pe^{rt} = 2000e^{.09(40)} \approx \$73,196.47$$

Historically, the stock market has earned an average rate of return of 9%.

### The number $e$ .

The number  $e$  is an irrational number and is approximately equal to 2.7183. The importance of the number  $e$  and the function  $y = e^x$  will not become evident until we study calculus.

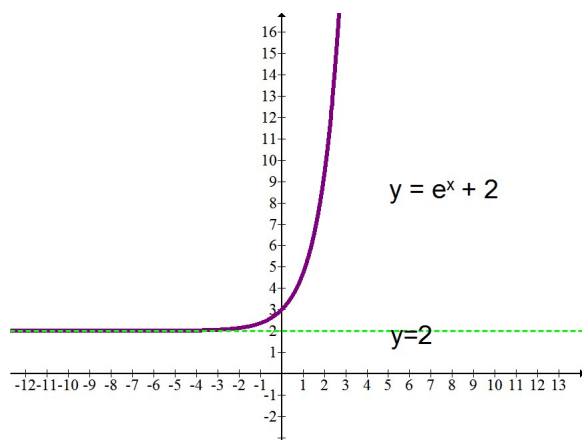
The graphs of  $y = 2^x$  and  $y = e^x$  are shown on the same plane. They really are not that different.



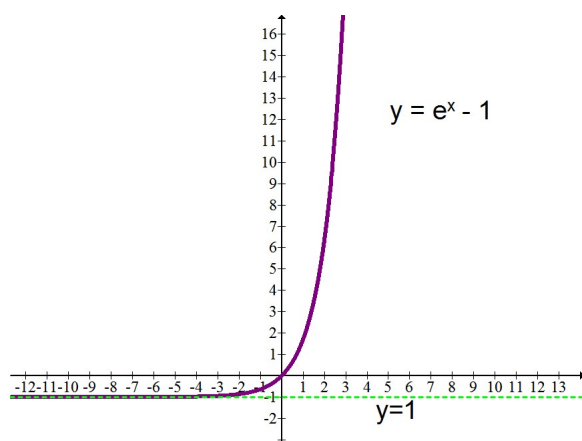
In the following example, we sketch transformations of the function  $y = e^x$ .

**Example** Sketch the graph.

1.  $y = e^x + 2$



2.  $y = e^{-x} - 1$



3.  $y = 5 \cdot e^x$

