4.3 Laws of Logarithms

In this section we talk about some of the special properties of logarithms.

Laws of Logarithms. For $A > 0$ and $B > 0$ and r any real number,

1. $\log_a(AB) = \log_a(A) + \log_a(B)$
2. $\log_a(A/B) = \log_a(A) - \log_a(B)$
3. $\log_a(A^r) = r\log_a(A)$

Let's prove the first statement. Let $x = \log_a(A)$ and $y = \log_a(B)$. Then if we convert these to exponential equations we get $a^x = A$ and $a^y = B$.

It follows that $AB = a^x \cdot a^y = a^{x+y}$.

If we convert this to logarithmic form, we get

$$\log_a(AB) = x + y$$

But substituting back in $x = \log_a A$ and $y = \log_a B$ gives

$$\log_a(AB) = \log_a(A) + \log_a(B) \qquad \square$$

Try proving the other statements on your own!

Example Write each expression as a single logarithm.

1. $\log_3(x) + \log_3(6)$

   SOLUTION $\log_3 x + \log_3 6 = \log_3 6x$

2. $\ln 3 + 2\ln x - 7\ln y$

SOLUTION

$$
\begin{aligned}
&\ln 3 + 2\ln x - 7\ln y \\
&\quad = \ln 3 + \ln x^2 - \ln y^7 \\
&\quad = \ln(3x^2) - \ln y^7 \\
&\quad = \ln\left(\frac{3x^2}{y^7}\right)
\end{aligned}
$$

3. $3\ln 2 - \dfrac{1}{2}\ln(x+1) - 4\ln(x+3)$

   SOLUTION

$$
\begin{aligned}
&3\ln 2 - \frac{1}{2}\ln(x+1) - 4\ln(x+3) \\
&\quad = \ln 2^3 - \ln(x+1)^{1/2} - \ln(x+3)^4 \\
&\quad = \ln 8 - \left(\ln\sqrt{x+1} + \ln(x+3)^4\right) \\
&\quad = \ln 8 - \ln\left(\sqrt{x+1}\cdot(x+3)^4\right) \\
&\quad = \ln\left(\frac{8}{\sqrt{x+1}\cdot(x+3)^4}\right)
\end{aligned}
$$

   Notice that the terms with a negative in front end up in the denominator, while the terms with a positive in front end up in the numerator.

Example Rewrite each expression using a sum or multiple of logarithms.

1. $\ln\left(\dfrac{x^2}{y^4\sqrt{z}}\right)$

   SOLUTION

$$
\begin{aligned}
&\ln\left(\frac{x^2}{y^4\sqrt{z}}\right) \\
&\quad = \ln x^2 - \ln y^4\sqrt{z} \\
&\quad = \ln x^2 - \left(\ln y^4 + \ln z^{1/2}\right) \\
&\quad = 2\ln x - 4\ln y - \frac{1}{2}\ln z
\end{aligned}
$$

Notice that the terms in the numerator end up with a positive in front, while the terms in the numerator end up with a negative in front. After you do a few of these, you can start skipping steps and go right to the answer.

2. $\log\left(\dfrac{(x-1)^2}{x^{3/2}(x+1)^3}\right)$

   SOLUTION $2\log(x-1) - \frac{3}{2}\log x - 3\log(x+1)$

The inverse of the exponential function $f(x) = a^x$ is $f^{-1}(x) = \log_a x$. Inverse functions have the cancelation property, that is, $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. Therefore, we can write down the following cancellation property for exponential and logarithmic functions.

Cancelation Properties. If $a > 0$ and $a \neq 1$, then

1. $\log_a(a^x) = x$ for any real number x.
2. $a^{\log_a(x)} = x$ for $x > 0$.

Example Simplify.

1. $e^{\ln x^2}$

   SOLUTION x^2

2. $\ln e^{2x+5}$

   SOLUTION $2x + 5$

We have some other properties of logarithms that follow from their definition.

- $\log_a a = 1$

  We can see this is true by rewriting the equation in exponential form. We get $a^1 = a$.

- $\log_a 1 = 0$

  We can see this is true by rewriting the equation in exponential form. We get $a^0 = 1$.

Suppose that I want to use a calculator to find a decimal approximation for $\log_2 5$. I see that there is no button for log base 2. So what I need to do is rewrite the expression in terms of either the natural log or the common log. To do this, I will use the Change of Base Formula.

Change of Base Formula

$$\log_a(M) = \frac{\log_b(M)}{\log_b(a)}$$

PROOF Let $y = \log_a M$. We can rewrite this as a logarithmic equation.

$$a^y = M$$

If we apply log base b to both sides, we get

$$\log_b a^y = \log_b M$$

Using the third law of logarithms, we get

$$y \log_b a = \log_b M,$$

that is,

$$y = \frac{\log_b B}{\log_b a}$$

Substituting back in for y gives

$$\log_a M = \frac{\log_b B}{\log_b a} \qquad \square$$

Example Write the expression $\log_2 5$ in terms of natural logs.

SOLUTION $\log_2 5 = \dfrac{\ln 5}{\ln 2}$

Example Write the expression $log_3 7$ in terms of common logs.

SOLUTION $\log_3 7 = \dfrac{\log 7}{\log 3}$