2.6 Mathematical Induction

A **statement** is a sentence or equation that is either true or false.

Example The following statement involving positive integers is true.

$$S_n : \quad 1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$$

It is easy to verify the statement is true for particular values of n, but this does not mean that it is true for all n. How can we prove that this is true for all n?

The Principle of Mathematical Induction: Let S_n be a statement for every positive integer n. If

1. S_1 is true, and
2. the truth of S_k implies the truth of S_{k+1} for every positive integer k

then S_n is true for every positive integer n.

Example Use mathematical induction to prove the following statements.

1. $S_n : \quad 1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$

   PROOF

   - Show that S_1 is true. When $n = 1$, we have $1 = 1^2$ which is certainly true.
   - Show that if S_k is true, then S_{k+1} is true.

   $$\begin{aligned}
   \text{Assume } S_k : \quad & 1 + 3 + 5 + 7 + \ldots + (2k - 1) &&= k^2 \\
   & +(2(k+1) - 1) && \quad +(2(k+1) - 1) \\
   & 1 + 3 + 5 + 7 + \ldots + (2k - 1) + (2(k+1) - 1) &&= k^2 + (2(k+1) - 1) \\
   & 1 + 3 + 5 + 7 + \ldots + (2k - 1) + (2(k+1) - 1) &&= k^2 + 2k + 2 - 1 \\
   & 1 + 3 + 5 + 7 + \ldots + (2k - 1) + (2(k+1) - 1) &&= k^2 + 2k + 1 \\
   S_{k+1} \quad & 1 + 3 + 5 + 7 + \ldots + (2k - 1) + (2(k+1) - 1) &&= (k+1)^2
   \end{aligned}$$

   This shows that if S_k is true, then S_{k+1} is true. □

2. $S_n = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ Proof

   - Show that S_1 is true. When $n = 1$, we have

$$\begin{aligned} 1^2 &= \frac{1(1+1)(2\cdot 1+1)}{6} \\ 1 &= \frac{1\cdot 2\cdot 3}{6} \end{aligned}$$

     which is true.

   - Show that if S_k is true, then S_{k+1} is true.

$$\begin{aligned}
\text{Assume } S_k : \quad 1^2 + 2^2 + 3^2 + \ldots + k^2 &= \frac{k(k+1)(2k+1)}{6} \\
+(k+1)^2 \quad & \quad +(k+1)^2 \\
1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \\
1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 &= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\
1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 &= \frac{(k+1)(2k^2 + k + 6k + 6)}{6} \\
1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 &= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 &= \frac{(k+1)(2k+3)(k+2)}{6} \\
S_{k+1} \quad 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}
\end{aligned}$$

     This shows that if S_k is true, then S_{k+1} is true. □

3. $W_n :\ 3^n < (n+2)!$ Proof

   - Show that W_1 is true. When $n = 1$, we have

$$3^1 < (1+2)!$$

$$\begin{aligned} 3 &< 3! \\ 3 &< 3 \cdot 2 \cdot 1 = 6 \end{aligned}$$

which is true.

- Show that if W_k is true, then W_{k+1} is true.

$$\begin{aligned} \text{Assume } W_k: \quad 3^k &< (k+2)! \\ 3^k &< (k+2)(k+1)k(k-1)\cdots 3 \cdot 2 \cdot 1 \end{aligned}$$

multiply both sides by 3

$$\begin{aligned} 3 \cdot 3^k &< 3(k+2)(k+1)k(k-1)\cdots 3 \cdot 2 \cdot 1 \\ 3^{k+1} &< 3(k+2)(k+1)k(k-1)\cdots 3 \cdot 2 \cdot 1 \\ \text{but} \quad 3 &< k+3 \\ \text{so} \quad 3^{k+1} &< (k+3)(k+2)(k+1)k(k-1)\cdots 3 \cdot 2 \cdot 1 \\ 3^{k+1} &< (k+3)! \end{aligned}$$

This shows that if W_k is true, then W_{k+1} is true. □