2.5 Inverse Functions

One-to-one Property

A function f is **one-to-one** if and only if each element in the range has a unique corresponding element in the domain. In other words, a function f is **one-to-one** if and only if for numbers a and b in the domain of f, we have

$$f(a) = f(b) \text{ implies } a = b.$$

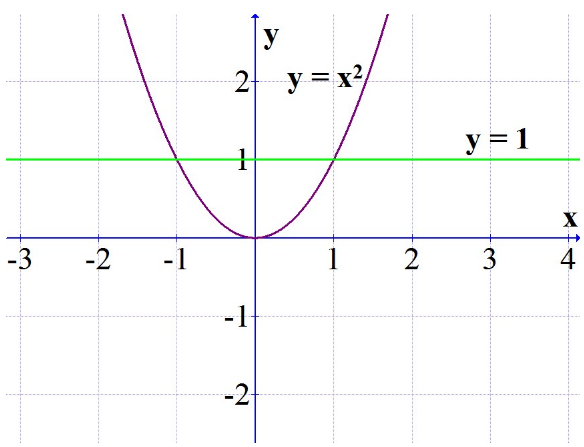
Example Show that $f(x) = x^3$ is one-to-one.

SOLUTION If $f(a) = f(b)$, then $a^3 = b^3$. This implies that $a = b$. Therefore, $f(x) = x^3$ is one-to-one.

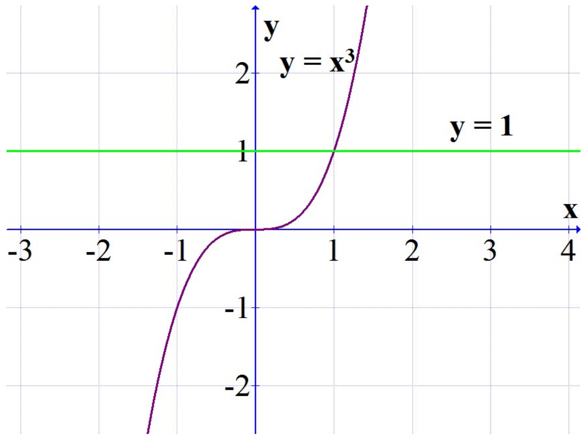
Example Show that $g(x) = x^2$ is not one-to-one.

SOLUTION We have $g(1) = (1)^2 = 1$ and $g(-1) = (-1)^2 = 1$. Therefore, $g(-1) = g(1)$ but, of course, $-1 \neq 1$. Therefore $g(x) = x^2$ is not one-to-one.

The graph of $y = x^2$ is shown below. The horizontal line $y = 1$ intersects the graph of $y = x^2$ at two points, $(1, 1)$ and $(-1, 1)$. We see that the y-value $y = 1$ corresponds to two different x-values, $x = 1$ or $x = -1$. There is not a one-to-one correspondence between the elements in the domain and range of $y = x^2$.



The graph of $y = x^3$ is shown below. If we draw any horizontal line $y = c$, then line will intersect the graph of $y = x^3$ at at most one point. This means that each y-value in the range corresponds to one and only one x-value in the domain. Therefore, the function $y = x^3$ is one-to-one.



The Horizontal Line Test: If each horizontal line crosses the graph of a function at no more than one point, then the function is one-to-one.

Example Determine whether a function f is one-to-one using the horizontal line test. Determine whether a function is one-to-one using the definition of one-to-one.

Inverse Functions

If f is a one-to-one function then we define the **inverse function** as the function f^{-1} such that

$$y = f^{-1}(x) \text{ is equivalent to } x = f(y)$$

Be Careful: The inverse function of f is written as f^{-1}. However, we should note that $f^{-1}(x)$ is not the same as $y = \dfrac{1}{f(x)}$!

The Switch and Solve Method

The inverse function of $y = f(x)$ is $y = f^{-1}(x)$, which in turn is equivalent to $x = f(y)$. So we see that to find the inverse of a one-to-one function, we interchange the x and y variables. After we interchange the x and y variables, we then solve for y. We call this the "Switch and Solve Method."

Example Find f^{-1} by using the Switch and Solve Method.

1. $f(x) = 4x - 1$

   Solution

$$\begin{aligned} y &= 4x - 1 \\ x &= 4y - 1 \quad \longleftarrow \text{ We switch the variables } x \text{ and } y \\ x + 1 &= 4y \quad \longleftarrow \text{ We now solve for } y \\ y &= \frac{x+1}{4} \\ f^{-1}(x) &= \frac{1}{4}x + \frac{1}{4} \quad \square \end{aligned}$$

2. $f(x) = \dfrac{2x+1}{x-3}$.

   Solution

$$\begin{aligned} y &= \frac{2x+1}{x-3} \\ x &= \frac{2y+1}{y-3} \quad \longleftarrow \text{ We switch the variables } x \text{ and } y \\ x(y-3) &= 2y + 1 \quad \longleftarrow \text{ We now solve for } y \\ xy - 3x &= 2y + 1 \\ xy - 2y &= 3x + 1 \\ y(x-2) &= 3x + 1 \\ y &= \frac{3x+1}{x-2} \\ f^{-1}(x) &= \frac{3x+1}{x-2} \quad \square \end{aligned}$$

Cancellation Property of Inverse Functions

Two functions f and g are inverses of each other if and only if

$$f(g(y)) = y \text{ and } g(f(x)) = x.$$

Example Let $f(x) = x^3 - 1$ and $g(x) = \sqrt[3]{x+1}$. Find $f(g(x))$ and $g(f(x))$. Are the two inverse functions of each other?

Solution

$$\begin{aligned} f(g(x) &= (g(x))^3 - 1 \\ &= (\sqrt[3]{x+1})^3 - 1 \\ &= (x+1) - 1 = x \end{aligned}$$

$$\begin{aligned} g(f(x)) &= \sqrt[3]{f(x)+1} \\ &= \sqrt[3]{(x^3-1)+1} \\ &= \sqrt[3]{x^3} = x \end{aligned}$$

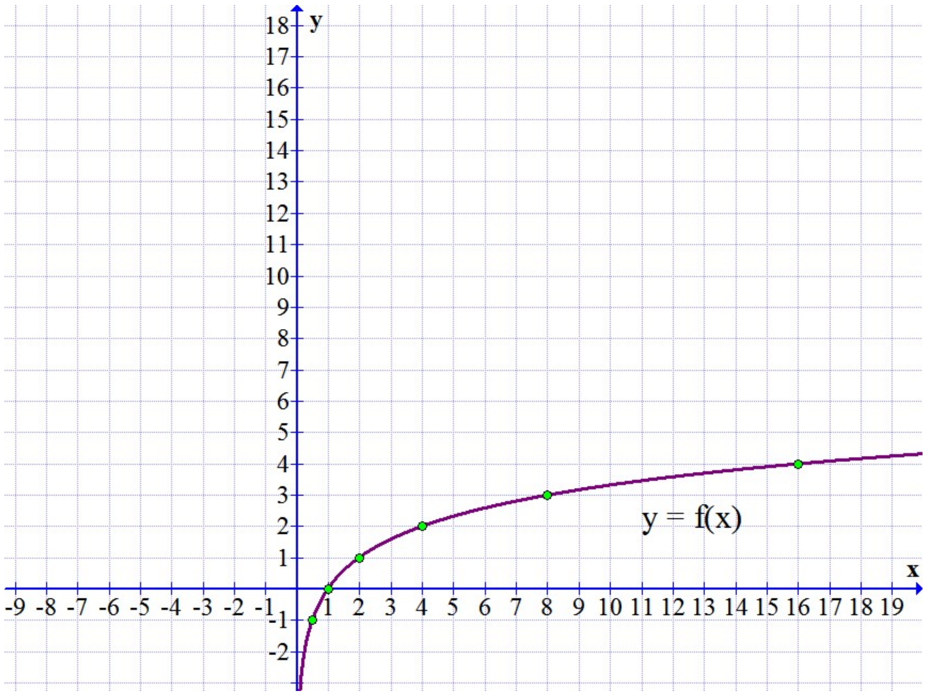
We see that $f(g(x)) = x$ and $g(f(x)) = x$. Therefore f and g are inverse functions of each other. □

The Graph of the Inverse Function

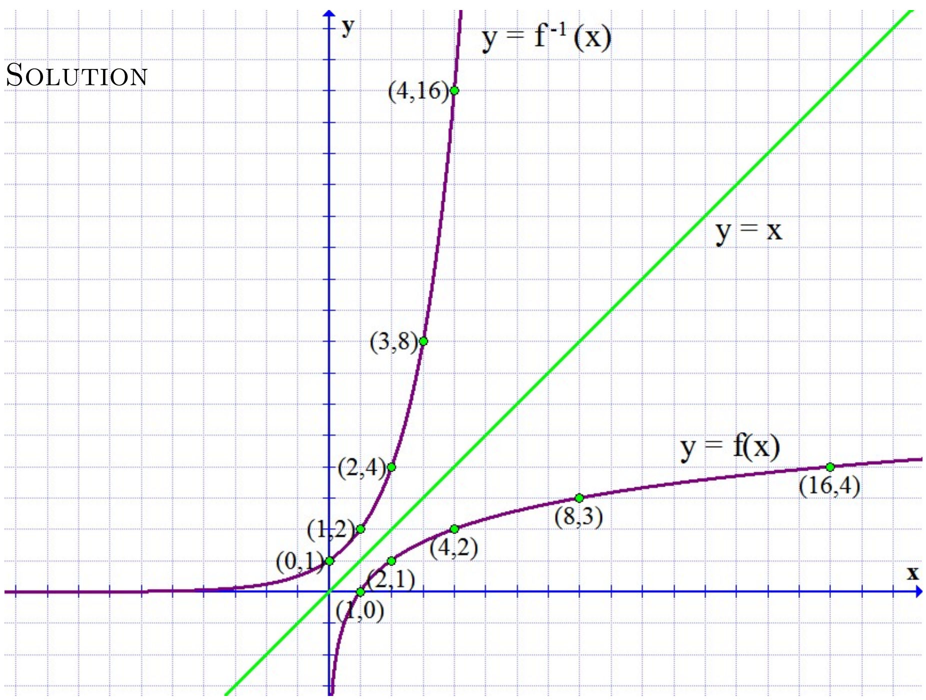
The inverse function of $y = f(x)$ is the function $y = f^{-1}(x)$ which is equivalent to $x = f(y)$. So we see that the inverse function is found by switching the x and y variables.

It follows that to sketch the graph of the inverse function f^{-1} we switch the x and y coordinates from the graph of f. This is equivalent to reflecting the graph of f about the line $y = x$. The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

Example The graph of a one-to-one function f is shown below. On the same plane, sketch the graph of its inverse f^{-1}.


y
x
y = f(x)
18
17
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
-1
-2
-9 -8 -7 -6 -5 -4 -3 -2 -1 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19

SOLUTION


y
x
y = f⁻¹(x)
y = x
y = f(x)
(4,16)
(3,8)
(2,4)
(1,2)
(0,1)
(16,4)
(8,3)
(4,2)
(2,1)
(1,0)