9.5 Determinants and Cramer's Rule

Any array of four numbers

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is called a 2×2 **matrix** (read **"two by two matrix"**). For instance

$$A = \begin{pmatrix} 2 & -3 \\ 5 & 4 \end{pmatrix}$$

is a 2×2 matrix. The four numbers, a, b, c, and d are called the **elements** of the matrix.

The **determinant** of a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is defined as the number $ad - bc$. We write

$$\det\, A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \tag{1}$$

We call the determinant of a 2×2 matrix a **second order determinant**. Notice a vertical bar is used when we write the determinant.

Example. Find the determinant of the matrix $\mathbf{A} = \begin{bmatrix} 2 & -3 \\ 5 & 4 \end{bmatrix}$.

Solution.

$$\det \mathbf{A} = \begin{vmatrix} 2 & -3 \\ 5 & 4 \end{vmatrix} = (2)(4) - (-3)(5) = 23.$$

□

We now use the determinant to derive a set of formulas for the solution of a system of two linear equations in two unknowns.

Consider the system

$$ax + by = u, \qquad (2)$$
$$cx + dy = v.$$

To solve the system, we multiply the first equation by d and the second equation by b, and subtract.

$$\begin{array}{rrcr} adx & +bdy & = & du, \\ -(bcx & +bdy) & = & -bv \\ \hline adx - bcx & & = & du - bv. \end{array}$$

The last line gives us an expression for x,

$$(ad - bc)x = du - bv$$
$$x = \frac{du - bv}{ad - bc}. \qquad (3)$$

Eliminating x instead of y from the equations gives us

$$y = \frac{av - cu}{ad - bc}. \qquad (4)$$

Observe that the solutions for x and y are fractions of expressions which are themselves determinants. We can write the expressions for x and y in the form

$$x = \frac{\begin{vmatrix} u & b \\ v & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & u \\ c & v \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}. \qquad (5)$$

Note how the column

$$\begin{pmatrix} u \\ v \end{pmatrix}$$

occurs as the first column in the numerator for x, and how it occurs in the second column in the numerator for y. In both cases, the denominator is the determinant $|A| = ad - bc$.

We now try to write the formulas for the solution of x and y in an even more concise form. Given a linear system of equations with two unknowns,

$$\begin{aligned} ax + by &= u, \\ cx + dy &= v, \end{aligned} \tag{6}$$

we can write the coefficients of x and y as a matrix, called the **coefficient matrix,**

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The numbers to the right of the equal sign can be put together in column vector called the **load vector**,

$$\begin{pmatrix} u \\ v \end{pmatrix}.$$

We define the matrix A_x as the matrix obtained by replacing the first column of A by the the load vector,

$$A_x = \begin{pmatrix} u & b \\ v & d \end{pmatrix}.$$

We define the matrix A_y as the matrix obtained by replacing the first column of A by the the load vector,

$$A_y = \begin{pmatrix} a & u \\ c & v \end{pmatrix}.$$

The formula for the solutions of x and y, given by (5), can now be written in terms of the determinants matrices A, A_x, and A_y,

$$x = \frac{\begin{vmatrix} u & b \\ v & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{|A_x|}{|A|}, \qquad y = \frac{\begin{vmatrix} a & u \\ c & v \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{|A_y|}{|A|}. \tag{7}$$

The formulas (7) are called **Cramer's rule.**

Notice that the solution is given by fractions with the determinant of the coefficient matrix A in the denominator. Therefore the solution to a linear system given by (2) exists if and only if the determinant of the coefficient matrix is not equal to zero.

Example. Use Cramer's rule to solve the following system of equations.

$$\begin{aligned} 2x - 3y &= 7, \\ 4x - y &= -5. \end{aligned} \tag{8}$$

Solution. We first write out the coefficient matrix A and find its determinant,

$$A = \begin{pmatrix} 2 & -3 \\ 4 & -1 \end{pmatrix}, \qquad |A| = (2)(-1) - (-3)(4) = 10.$$

The numbers that are to the right of the equal sign of the system (8) form the load vector

$$\begin{pmatrix} 7 \\ -5 \end{pmatrix}.$$

The matrix A_x is formed by replacing the first column of A by the load vector. We write out A_x and find its determinant,

$$A_x = \begin{pmatrix} 7 & -3 \\ -5 & -1 \end{pmatrix}, \quad |A_x| = (7)(-1) - (-3)(-5) = -22.$$

Likewise, the matrix A_y is formed by replacing the second column of A by the load vector. We write out A_y and find its determinant,

$$A_y = \begin{pmatrix} 2 & 7 \\ 4 & -5 \end{pmatrix}, \quad |A_y| = (2)(-5) - (7)(4) = -38.$$

We find the solution for x and y by using the formulas given by (7),

$$x = \frac{|A_x|}{|A|} = \frac{-22}{10} = -\frac{11}{5}, \quad y = \frac{|A_y|}{|A|} = \frac{-38}{10} = -\frac{19}{5}.$$

We can check our answer by plugging the values for x and y back in to the original system of equations (8) and verifying that the equality holds. In this case, checking is a little bit of trouble because of the fractions involved.

$$\begin{aligned} 2(-11/5) - 3(-19/5) &= 7, \\ 4(-11/5) - (-19/5) &= -5. \end{aligned}$$

□

The Determinant of a 3 by 3 Matrix

An array of numbers

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \tag{9}$$

is called a 3×3 matrix.

The **minor** of an element is found by deleting the row and column in which the element belongs and then taking the determinant of the resulting matrix. For example, the minor of a_1 is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix},$$

the minor of b_1 is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix},$$

and the minor of c_1 is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

The value of a **third order determinant** is obtained by multiplying the elements of the first row by their respective minors and then summing with signs attached by the pattern $+ \; - \; +$. The determinant of a 3×3 matrix is defined as

$$\begin{aligned} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} &= a_1 \cdot \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} - a_2 \cdot \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + a_3 \cdot \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ &= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}. \end{aligned} \tag{10}$$

Example. Find the determinant of the matrix

$$A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 4 & -2 \\ 1 & 5 & 2 \end{pmatrix}.$$

Solution.

$$\begin{aligned} |A| &= \begin{vmatrix} 2 & 1 & 3 \\ -1 & 4 & -2 \\ 1 & 5 & 2 \end{vmatrix} \\ &= 2 \cdot \begin{vmatrix} 2 & 1 & 3 \\ -1 & 4 & -2 \\ 1 & 5 & 2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 2 & 1 & 3 \\ -1 & 4 & -2 \\ 1 & 5 & 2 \end{vmatrix} + 3 \cdot \begin{vmatrix} 2 & 1 & 3 \\ -1 & 4 & -2 \\ 1 & 5 & 2 \end{vmatrix} \\ &= 2 \cdot \begin{vmatrix} 4 & -2 \\ 5 & 2 \end{vmatrix} - 1 \cdot \begin{vmatrix} -1 & -2 \\ 1 & 2 \end{vmatrix} + 3 \cdot \begin{vmatrix} -1 & 4 \\ 1 & 5 \end{vmatrix} \\ &= 2((4)(2) - (-2)(5)) - 1((-1)(2) - (-2)(1)) + 3 \cdot ((-1)(5) - (4)(1)) \\ &= 2 \cdot (18) - 1 \cdot (0) + 3 \cdot (-9) = 9. \end{aligned}$$

□

In this definition of determinant, we are expanding across the first row, but in fact, it is possible to expand across any row or column to find the determinant. The sign pattern when expanding across the first row is $+ \ - \ +$, that is, we add the first term, subtract the second term and add the third term. If we expand across a different row or column, we should follow the pattern

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}.$$

We always start with addition in the upper left had corner. We then fill out the matrix in a checker board pattern of positive or negatives.

Example. Find the determinant of the matrix A given in the previous example by expanding across the *second row*.

Solution.

$$
\begin{aligned}
|A| &= \begin{vmatrix} 2 & 1 & 3 \\ -1 & 4 & -2 \\ 1 & 5 & 2 \end{vmatrix} \\
&= -(-1)\cdot \begin{vmatrix} 2 & 1 & 3 \\ -1 & 4 & -2 \\ 1 & 5 & 2 \end{vmatrix} + (4)\cdot \begin{vmatrix} 2 & 1 & 3 \\ -1 & 4 & -2 \\ 1 & 5 & 2 \end{vmatrix} - (-2)\cdot \begin{vmatrix} 2 & 1 & 3 \\ -1 & 4 & -2 \\ 1 & 5 & 2 \end{vmatrix} \\
&= -(-1)\cdot \begin{vmatrix} 1 & 3 \\ 5 & 2 \end{vmatrix} + (4)\cdot \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} - (-2)\cdot \begin{vmatrix} 2 & 1 \\ 1 & 5 \end{vmatrix} \\
&= -(-1)((1)(2)-(3)(5)) + (4)((2)(2)-(3)(1)) - (-2)\cdot((2)(5)-(1)(1)) \\
&= 1\cdot(-13) + 4\cdot(1) + 2\cdot(9) = 9.
\end{aligned}
$$

□

Cramer's Rule

We now give a formula for the solution to a system of linear equations with three unknowns. The formula is called Cramer's rule, and it is the same formula that was given for the solution of systems of linear equations with two unknowns, except that there is now one more variable. In fact, Cramer's rule can be generalized to linear systems of equations with n unknowns.

Consider the liner system of equations

$$
\begin{aligned}
a_1x + b_1y + c_1z &= d_1 \\
a_2x + b_2y + c_2z &= d_2 \\
a_3x + b_3y + c_3z &= d_3.
\end{aligned} \tag{11}
$$

The coefficients of x, y, and z can be written as a coefficient matrix,

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}, \tag{12}$$

and the numbers to the right of the equal sign can be written as the load vector

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \tag{13}$$

The matrix A_x is formed by replacing the first column of A by the load vector, the matrix A_y is formed by replacing the second column of A by the load vector, and the matrix A_z is formed by replacing the third column of A by the load vector. That is,

$$A_x = \begin{pmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{pmatrix}, \; A_y = \begin{pmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{pmatrix}, \; A_z = \begin{pmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{pmatrix}. \tag{14}$$

Then the solution to the linear system (11) is given by

$$x = \frac{|A_x|}{|A|}, \quad y = \frac{|A_y|}{|A|}, \quad z = \frac{|A_z|}{|A|}. \tag{15}$$

Again we point out that the solution is given by fractions with the determinant of the coefficient matrix A in the denominator. Therefore the solution to a linear system given by (11) exists if and only if the determinant of the coefficient matrix is not equal to zero.

Example. Find the solution to the given system of linear equations using Cramer's rule.

$$\begin{aligned} 2x + y + 3z &= 13 \\ -x + 4y - 2z &= 1 \\ x + 5y + 2z &= 17 \end{aligned} \tag{16}$$

Solution. The coefficient matrix is the same matrix A given in the previous two examples. In those examples, we showed that the determinant of A is equal to 9.

$$|A| = \begin{vmatrix} 2 & 1 & 3 \\ -1 & 4 & -2 \\ 1 & 5 & 2 \end{vmatrix} = 9.$$

The matrix A_x is obtained by replacing the first row of A by the load vector

$$\begin{pmatrix} 13 \\ 1 \\ 17 \end{pmatrix}.$$

We can find the determinant of A_x by expanding across the first row.

$$\begin{aligned}|A_x| =& \begin{vmatrix} 13 & 1 & 3 \\ 1 & 4 & -2 \\ 17 & 5 & 2 \end{vmatrix} \\ =& 13 \cdot \begin{vmatrix} 13 & 1 & 3 \\ 1 & 4 & -2 \\ 17 & 5 & 2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 13 & 1 & 3 \\ 1 & 4 & -2 \\ 17 & 5 & 2 \end{vmatrix} + 3 \cdot \begin{vmatrix} 13 & 1 & 3 \\ 1 & 4 & -2 \\ 17 & 5 & 2 \end{vmatrix} \\ =& 13 \cdot \begin{vmatrix} 4 & -2 \\ 5 & 2 \end{vmatrix} - 1 \cdot \begin{vmatrix} -1 & -2 \\ 17 & 2 \end{vmatrix} + 3 \cdot \begin{vmatrix} 1 & 4 \\ 17 & 5 \end{vmatrix} \\ =& 13((4)(2) - (-2)(5)) - 1((-1)(2) - (-2)(17)) + 3 \cdot ((1)(5) - (4)(17)) \\ &= 13 \cdot (18) - 1 \cdot (36) + 3 \cdot (-63) = 9. \end{aligned} \tag{17}$$

The matrix A_y is obtained by replacing the first row of A by the load vector

$$\begin{pmatrix} 13 \\ 1 \\ 17 \end{pmatrix}.$$

We can find the determinant of A_y by expanding across the first row.

$$\begin{aligned}|A_y| =& \begin{vmatrix} 2 & 13 & 3 \\ -1 & 1 & -2 \\ 1 & 17 & 2 \end{vmatrix} \\ =& 2 \cdot \begin{vmatrix} 2 & 13 & 3 \\ -1 & 1 & -2 \\ 1 & 17 & 2 \end{vmatrix} - 13 \cdot \begin{vmatrix} 2 & 13 & 3 \\ -1 & 1 & -2 \\ 1 & 17 & 2 \end{vmatrix} + 3 \cdot \begin{vmatrix} 2 & 13 & 3 \\ -1 & 1 & -2 \\ 1 & 17 & 2 \end{vmatrix} \\ =& 2 \cdot \begin{vmatrix} 1 & -2 \\ 17 & 2 \end{vmatrix} - 13 \cdot \begin{vmatrix} -1 & -2 \\ 1 & 2 \end{vmatrix} + 3 \cdot \begin{vmatrix} -1 & 4 \\ 1 & 17 \end{vmatrix} \\ =& 2((1)(2) - (-2)(17)) - 13((-1)(2) - (-2)(1)) + 3 \cdot ((-1)(17) - (1)(1)) \\ &= 2 \cdot (36) - 1 \cdot (0) + 3 \cdot (-18) = 18. \end{aligned} \tag{18}$$

The matrix A_z is obtained by replacing the first row of A by the load vector

$$\begin{pmatrix} 13 \\ 1 \\ 17 \end{pmatrix}.$$

We can find the determinant of A_z by expanding across the first row.

$$\begin{aligned}
|A_z| =& \begin{vmatrix} 2 & 1 & 13 \\ -1 & 4 & 1 \\ 1 & 5 & 17 \end{vmatrix} \\
=& 2 \cdot \begin{vmatrix} 2 & 1 & 13 \\ -1 & 4 & 1 \\ 1 & 5 & 17 \end{vmatrix} - 1 \cdot \begin{vmatrix} 2 & 1 & 13 \\ -1 & 4 & 1 \\ 1 & 5 & 17 \end{vmatrix} + 13 \cdot \begin{vmatrix} 2 & 1 & 13 \\ -1 & 4 & 1 \\ 1 & 5 & 17 \end{vmatrix} \\
=& 2 \cdot \begin{vmatrix} 4 & 1 \\ 5 & 17 \end{vmatrix} - 1 \cdot \begin{vmatrix} -1 & 1 \\ 1 & 17 \end{vmatrix} + 13 \cdot \begin{vmatrix} -1 & 4 \\ 1 & 5 \end{vmatrix} \\
=& 2\left((4)(17) - (5)(1)\right) - 1\left((-1)(17) - (1)(1)\right) + 13 \cdot \left((-1)(5) - (1)(4)\right) \\
=& 2 \cdot (63) - 1 \cdot (-18) + 13 \cdot (-9) = 27.
\end{aligned} \tag{19}$$

We can now use the formula for the solution given by Cramer's rule (15),

$$x = \frac{|A_x|}{|A|} = \frac{9}{9} = 1, \quad y = \frac{|A_y|}{|A|} = \frac{18}{9} = 2, \quad z = \frac{|A_z|}{|A|} = \frac{27}{9} = 3. \tag{20}$$

We can check that the solution is correct by plugging these values back into the original system of equations and verifying that equality holds.

$$\begin{aligned}
2(1) + (2) + 3(3) &= 13 \\
-(1) + 4(2) - 2(3) &= 1 \\
(1) + 5(2) + 2(3) &= 17
\end{aligned} \tag{21}$$

□