3.6 Rational Functions

Introduction

A rational function is a fraction of two polynomial functions.

Some examples of rational functions are

1. $y = \dfrac{1}{x}$
2. $y = \dfrac{x^2 + 6x + 5}{x^2 + 5x + 6}$

The Domain of a Rational Function

The domain of a rational function is the set of all real numbers except those numbers that will result in a zero denominator.

Example. State the domain of the following function.

$$y = \frac{x^2 + 6x + 5}{x^+5x + 6}$$

SOLUTION The first thing to do is, if possible, factor the polynomials in the rational function.

$$y = \frac{x^2 + 6x + 5}{x^2 + 5x + 6} = \frac{(x + 1)(x + 5)}{(x + 2)(x + 3)}$$

The function is undefined when the denominator is equal to 0.

$$(x + 2)(x + 3) = 0$$

Therefore the numbers where f is undefined are $x = -2$ and $x = -3$. We have

$$\begin{aligned} \text{Dom } f &= \{x \in \mathbf{R} | x \neq -2 \text{ and } x \neq -3\} \\ &= \text{All real numbers except } x = 2 \text{ and } x = 3 \end{aligned}$$

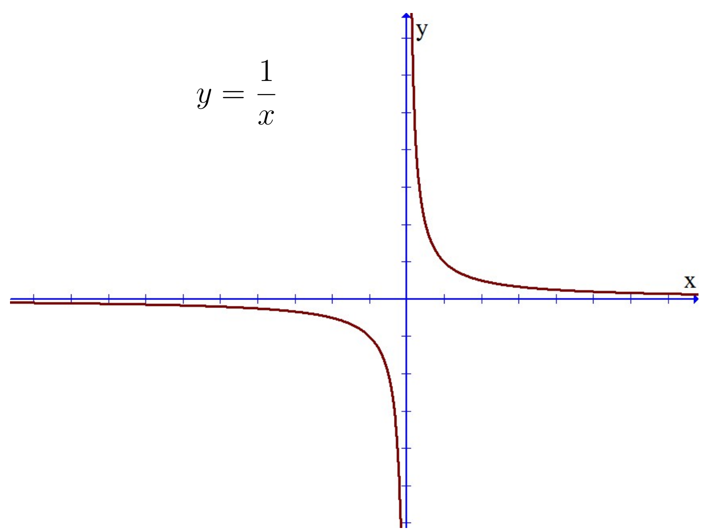
Vertical Asymptotes

What is the behavior of a rational function when the x-coordinate is near one of these undefined numbers?

Let's study the function $f(x) = \dfrac{1}{x}$. The domain of this function is all real numbers, $x \neq 0$.

Below is a table of values and the graph.

x	$f(x) = \frac{1}{x}$
10	1/10
5	1/5
2	1/2
1	1
1/2	2
1/4	4
1/10	10
1/100	100
1/1000	100
−10	−1/10
−5	−1/5
−2	−1/2
−1	−1
−1/2	−2
−1/4	−4
−1/10	−10
−1/100	−100
−1/1000	−100



As x approaches the undefined number, 0, from the right, we see that the y-coordinate goes up through the roof. That is,

$$\text{As } x \to 0^+, \quad y \to \infty$$

As x approaches the undefined number, 0, from the left, we see that the

y-coordinate goes up through the floor. That is,

$$\text{As } x \to 0^-, \quad y \to -\infty$$

The line $x = 0$ is called a vertical asymptote of the function $y = 1/x$. As the graph approaches the vertical asymptote, the graph will either go through the roof, that is, $f(x) \to \infty$, or the graph goes through the floor, that is, $f(x) \to -\infty$.

The way to find out whether the graph goes through the floor or through the roof is to plot points, but you do not need to plot that many points. You only need to plot enough points to determine whether the function is positive or negative near the vertical asymptote.

Vertical asymptotes always occur at undefined numbers. However, some undefined numbers do not give vertical asymptotes.

Example. Find the domain of the function $f(x) = \dfrac{x^2 - 1}{x - 1}$ and sketch the graph.

SOLUTION

We find the undefined numbers of the function. The denominator equals zero when

$$x - 1 = 0$$

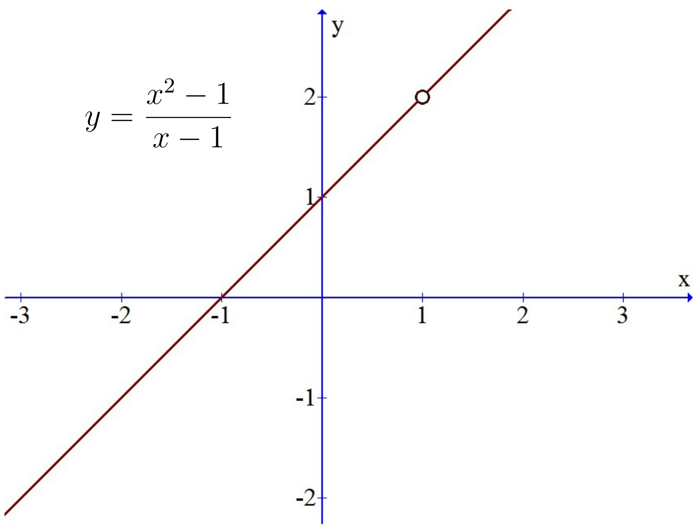
$$x = 1$$

The domain is the set of all real numbers $x \neq 1$.

To sketch the graph, let's first factor the polynomial in the numerator of the function.

$$f(x) = \frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1} = x + 1, \quad x \neq 1$$

The graph of this function is the same as the graph of $y = x + 1$ except that it has a hole at the point $(1, 2)$.



This rational function is undefined at $x = 1$, but it does not have a vertical asymptote.

Zeros of Rational Functions

To find the zeros of a rational function, set the numerator equal to zero and then solve for x. It is a good idea to find the domain of the function first, because it is possible that a number will make both the numerator and the denominator equal to zero. In this case, the number is not in the domain of the function and is not a zero of the function.

Example. Find the zero of the rational function

$$y = \frac{x^2 - 5x - 6}{x^2 - 1}$$

SOLUTION Set the numerator equal to zero and solve for x.

$$\begin{aligned} x^2 - 5x - 6 &= 0 \\ (x-6)(x+1) &= 0 \\ x = 6, \quad x = -1 & \end{aligned}$$

Horizontal Asymptotes

Let's return to the graph of $y = 1/x$. We notice that as x becomes a large number, the graph of the function gets very close to the x-axis. It never touches the x-axis because the y-coordinate is never 0. This is because the numerator of the function $y = 1/x$ is never 0.

We can write

$$\text{As } x \to \infty, \quad y \to 0$$

and

$$\text{As } x \to -\infty, \quad y \to 0$$

The function $y = 1/x$ is said to have a **horizontal asymptote** at the line $y = 0$, that is, the x-axis.

To find the horizontal of a rational function, we will use the following principle.

> For large values of x, a polynomial behaves like its highest power term.

Example. Find the horizontal asymptote (if one exists) of the function

$$y = \frac{x^2 + 6x + 5}{x^2 + 5x + 6}$$

SOLUTION For finding the horizontal asymptote, factoring the polynomials in the function does not help us. Instead, we will use the principle given above.

$$\text{As } x \to \infty, \quad y = \frac{x^2 + 6x + 5}{x^2 + 5x + 6} \to \frac{x^2}{x^2} = 1$$

The horizontal asymptote is the line $y = 1$.

If a horizontal asymptote exists for a rational function, then we only need to look at the behavior as $x \to \infty$. We will get the same answer as $x \to -\infty$.

Example. Find the horizontal asymptote (if one exists) of the function

$$y = \frac{5x^2 + 6x + 5}{7x^2 + 5x + 6}$$

SOLUTION

$$\text{As } x \to \infty, \quad y = \frac{5x^2+6x+5}{7x^2+5x+6} \to \frac{5x^2}{7x^2} = \frac{5}{7}$$

The horizontal asymptote is the line $y = \frac{5}{7}$.

If the degree of the numerator equals the degree of the denominator of a rational function, then the function will have horizontal asymptote $y = \frac{a_n}{b_n}$, where a_n is the lead coefficient of the numerator, and b_n is the lead coefficient of the denominator.

Example. In this example, the degree of the denominator is greater than the degree of the numerator.

Find the horizontal asymptote (if one exists) of the function

$$y = \frac{5x^2+6x+5}{x^3+5x+6}$$

SOLUTION

$$\text{As } x \to \infty, \quad y = \frac{5x^2+6x+5}{x^3+5x+6} \to \frac{5x^2}{x^3} = \frac{5}{x} \to 0$$

The horizontal asymptote is the line $y = 0$, the x-axis

If the denominator of a rational function has higher degree than the numerator, then the function will have horizontal asymptote the x-axis.

Example. In this example, the degree of the numerator is greater than the degree of the denominator.

Find the horizontal asymptote (if one exists) of the function

$$y = \frac{5x^3+6x+5}{x^2+5x+6}$$

SOLUTION

$$\text{As } x \to \infty, \quad y = \frac{5x^3+6x+5}{x^2+5x+6} \to \frac{5x^3}{x^2} = 5x \to \infty$$

There is no horizontal asymptote. As $x \to \infty$, the graph of the function will go through the roof.

$$\text{As } x \to -\infty, \quad y = \frac{5x^3 + 6x + 5}{x^2 + 5x + 6} \to \frac{5x^3}{x^2} = 5x \to -\infty$$

For rational functions, if the numerator has higher degree than the denominator, then the function will not have a horizontal asymptote. For large values of x, it will behave like a polynomial. As x becomes large, the graph will either go to infinity (through the roof) or it will go to negative infinity (through the floor). As with a polynomial, it is necessary to check the behavior as x goes to both positive and negative infinity.

Oblique Asymptotes

If the degree of the numerator is one more than the degree of the denominator of a rational function, then there is no horizontal asymptote, but there is an oblique asymptote. Suppose we have a rational function $\frac{f(x)}{g(x)}$ where f and g are polynomials and $\deg f = \deg g + 1$. By the Euclidean algorithm, there exists polynomials q, r such that $\deg r < \deg g$ and such that

$$f(x) = q(x)g(x) + r(x)$$

Dividing both sides through by $g(x)$ gives

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$$

We have written the original rational function as the sum of a polynomial, $q(x)$, and a rational function, $\frac{r(x)}{g(x)}$, where the degree of the numerator is less than the degree of the denominator. The rational function $\frac{r(x)}{g(x)}$ has the x-axis, i.e. $y = 0$, as a horizontal asymptote because $\deg r < \deg g$. This means that as x becomes very large, $\frac{r(x)}{g(x)}$ becomes very small. The result is that the original rational function $\frac{f(x)}{g(x)}$ behaves like the polynomial $q(x)$ when x becomes large. Because the degree of f is one more than the degree of g, the polynomial $q(x)$ will have degree 1, and therefore it will be a line. The **oblique asymptote** is the line $y = q(x)$ guaranteed to exist by the Euclidean algorithm.

Example. Find the oblique asymptote of the rational function $y = \frac{x^2+3x+5}{x+1}$.

SOLUTION We use long division to divide $x^2 + 3x + 5$ by $x + 1$.

We see that

$$y = \frac{x^2 + 3x + 5}{x + 1} = x + 2 + \frac{3}{x + 1}$$

As x becomes very large, $\dfrac{3}{x+1}$ approaches zero, and the rational function approaches the oblique asymptote $y = x + 2$.

ANSWER: Oblique Asymptote $y = x + 2$.

Graphs of Rational Functions

The goal of this chapter is to be able to make a rough sketch of a rational function. Let's go straight to an example.

Example. Degree of Numerator = Degree of Denominator.

$$\text{Let } f(x) = \frac{x - 1}{x + 2}.$$

1. State the domain of f.

   SOLUTION We set the denominator equal to zero and solve for x.

$$\begin{aligned} x + 2 &= 0 \\ x &= -2 \end{aligned}$$

   The domain of f is all real numbers <u>except</u> $x = -2$.

   The line $x = -2$ will be a vertical asymptote.

2. Find the zeros of f.

   SOLUTION

   Now we set the numerator to 0 and solve for x.

$$\begin{aligned} x - 1 &= 0 \\ x &= 1 \end{aligned}$$

Therefore, the x-intercept is $(1, 0)$.

3. Find a horizontal asymptote, if it exists.

   SOLUTION

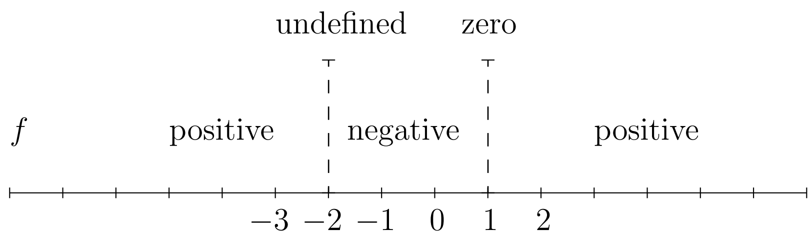
   $$\text{As } x \to \infty, \quad \frac{x-1}{x+2} \to \frac{x}{x} = 1$$

   The function f has horizontal asymptote $y = 1$.

4. Make a chart showing where the function is positive and negative.

   SOLUTION

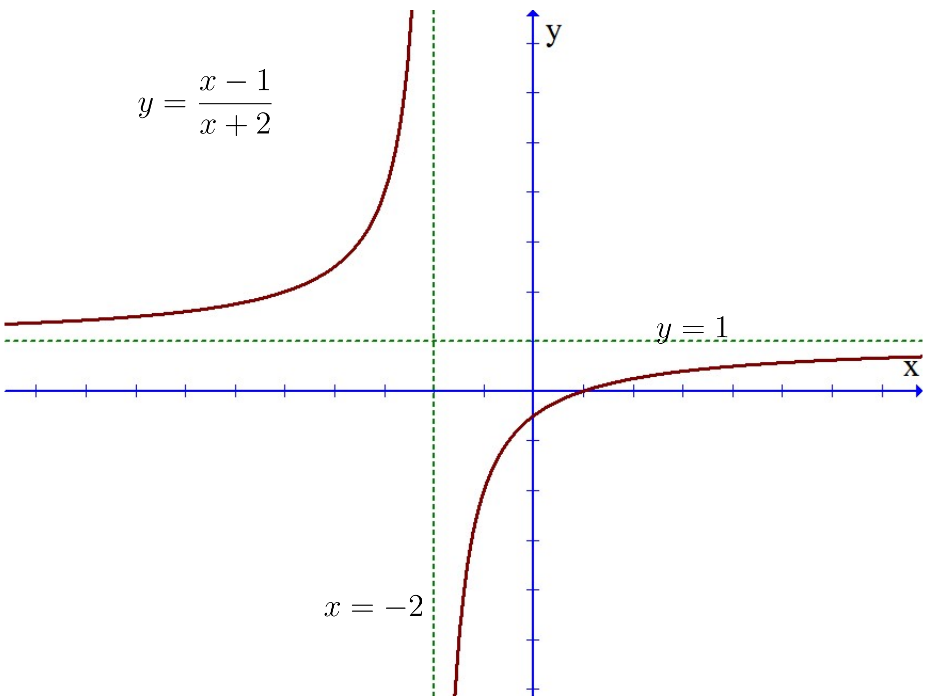
   The chart is really an number line. We break the number line into intervals whose endpoints are either zeros of the function or undefined numbers. A function will only change signs at either a zero or at an undefined number. We want to determine whether the function is positive or negative on these intervals. To do this, we can plot points.



x	$f(x) = \frac{x-1}{x+2}$	sign
-3	$\frac{-3-1}{-3+2} = \frac{-4}{-1} = 4$	positive
0	$\frac{0-1}{0+2} = \frac{-1}{2}$	negative
2	$\frac{2-1}{2+2} = \frac{1}{4}$	positive

5. Make a rough sketch of the graph.

   SOLUTION



Example. Degree of Numerator < Degree of Denominator.

$$\text{Let } f(x) = \frac{x-2}{x^2 - 3x - 4}.$$

1. State the domain of f.

   SOLUTION

$$f(x) = \frac{x-2}{(x-4)(x+1)}.$$

   f is undefined when $x = 4$ or $x = 1$. Therefore the domain of f is all real numbers, $x \neq 4$, $x \neq -1$.

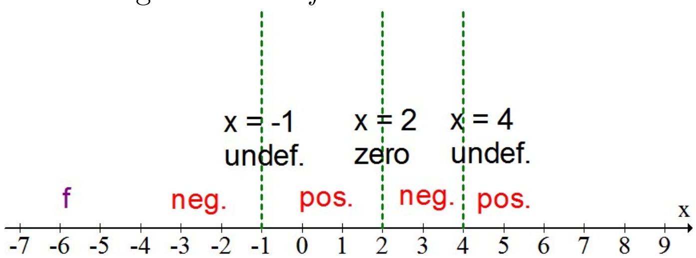
2. Find the zeros of f.

   SOLUTION $x = 2$

3. Find a horizontal asymptote, if it exists.

   SOLUTION As $x \to \infty$, $f(x) \to \frac{x}{x^2} \to 0$. So f has horizontal asymptote $y = 0$, that is, the x-axis.

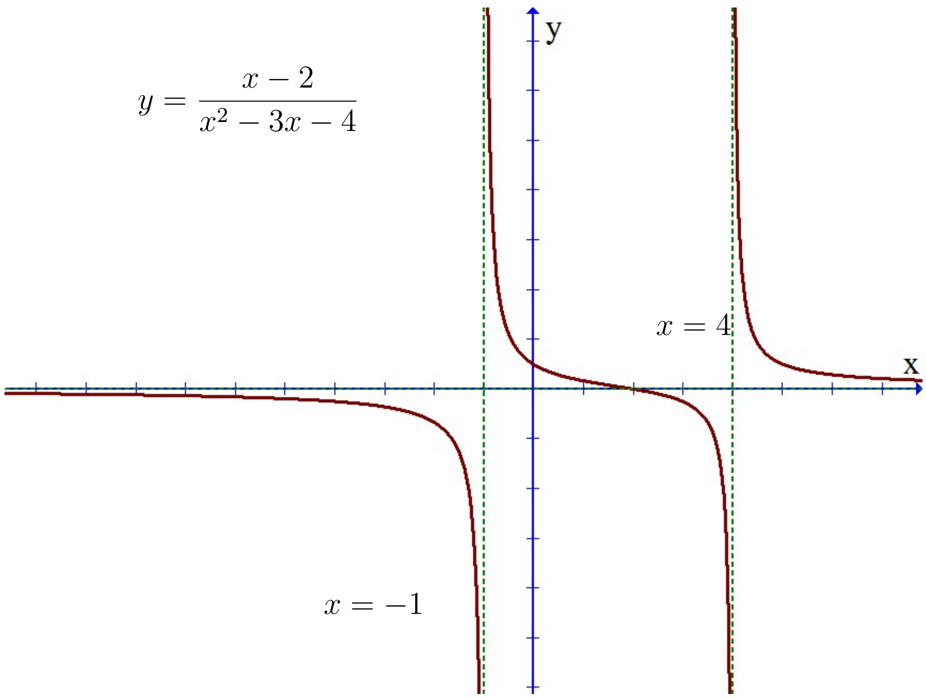
4. Make a sign chart for f.



x	$f(x) = \frac{x-2}{(x-4)(x+1)}$	*sign*
-2	$-2/3$	negative
0	$1/2$	positive
3	$-1/4$	negative
5	$1/2$	positive

5. Make a rough sketch of the graph.

   SOLUTION



Example. Degree of Numerator $= 1+$ **Degree of Denominator.**

$$\text{Let } f(x) = \frac{x^2 - 2x - 8}{x - 1}.$$

1. State the domain of f.

   SOLUTION The domain of f is all real numbers, $x \neq 1$.

2. Find the zeros of f.

   SOLUTION
   $$f(x) = \frac{(x-4)(x+2)}{x-1}.$$
   The zeros of f are $x = 4$ and $x = -2$.

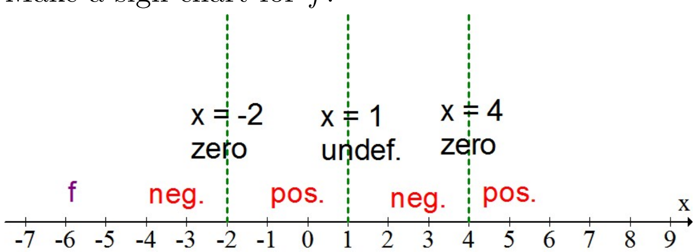
3. Find a horizontal asymptote or oblique asymptote of f.

   SOLUTION Because the degree of the numerator is one more than the degree of the denominator, f will have an oblique asymptote.

$$\begin{array}{r|l} & x-1 \quad \text{r. } -9 \\ \hline x-1 & x^2-2x-8 \\ & \underline{-(x^2-x)} \\ & \quad -x-8 \\ & \underline{-(-x+1)} \\ & \quad\quad -9 \end{array}$$

   $f(x) = x - 1 - \frac{9}{x-1}$. The oblique asymptote is the line $y = x - 1$.

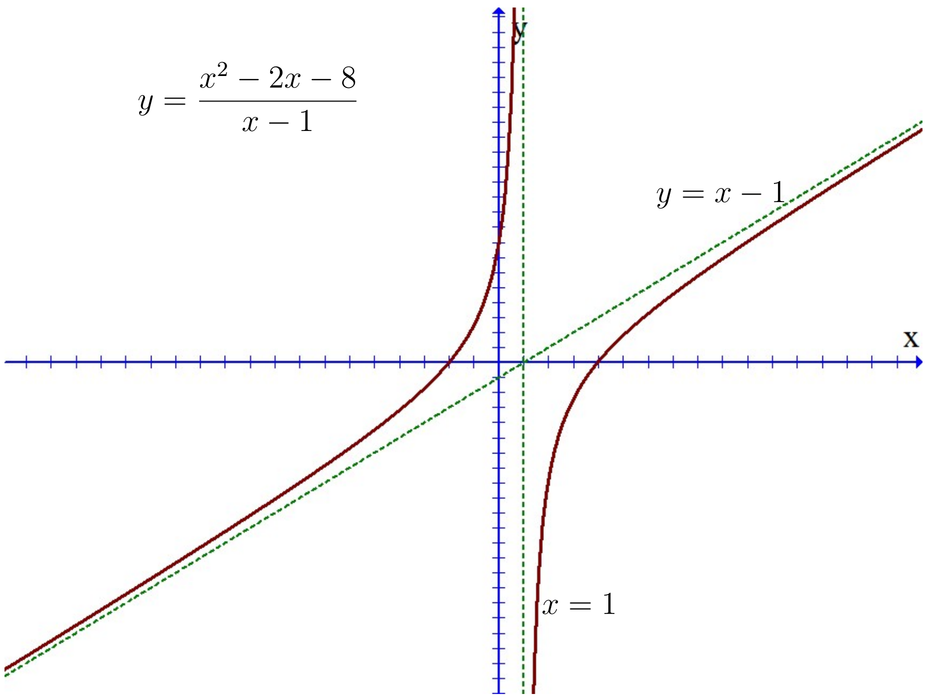
4. Make a sign chart for f.



x	$f(x) = \frac{(x-4)(x+2)}{x-1}$	*sign*
-3	$-7/4$	negative
0	8	positive
2	-8	negative
5	$7/4$	positive

5. Make a rough sketch of the graph.

   SOLUTION

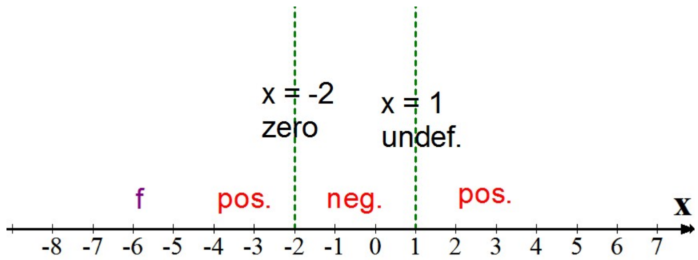


Rational Inequalities

Example. Solve the rational inequality.

$$\frac{x+2}{x-1} > 0$$

SOLUTION We make a sign chart for the function $f(x) = \frac{x+2}{x-1}$. There is a zero at $x = -2$ and a number $x = 1$ where f is undefined.



x	$f(x) = \frac{x+2}{x-2}$	
-3	$1/4$	positive
0	-2	negative
2	4	positive

We can read the sign chart to conclude that the function is positive on the intervals $(-\infty, -2) \cup (1, \infty)$.

Adding and Subtracting Rational Expressions

Example. Add the two rational expressions.

$$\frac{x+2}{x-3} + \frac{x-1}{x^2-5x+6}$$

SOLUTION First factor the denominator and then find a common denominator.

$$\begin{aligned}
\frac{x+2}{x-3} + \frac{x-1}{x^2-5x+6} &= \frac{x+2}{x-3} + \frac{x-1}{(x-3)(x-2)} \\
&= \frac{x+2}{x-3} \cdot \frac{x-2}{x-2} + \frac{x-1}{(x-3)(x-2)} \\
&= \frac{(x+2)(x-2)}{(x-3)(x-2)} + \frac{x-1}{(x-3)(x-2)} \\
&= \frac{x^2-4}{(x-3)(x-2)} + \frac{x-1}{(x-3)(x-2)}
\end{aligned}$$

$$= \frac{(x^2 - 4) + (x - 1))}{(x-3)(x-2)}$$
$$= \frac{x^2 + x - 5}{x^2 - 5x + 6}$$