11.1 Sequences

A **sequence** is an ordered list of numbers. For example, the even numbers,

$$2, 4, 6, 8, \ldots,$$

form a sequence.

An alternate definition is to say that a sequence is a function whose domain is the set of positive integers.

We are familiar, for example, with the function $f(x) = x^2$, where x is any real number. If we restrict the domain to the positive integers, $1, 2, 3, \ldots,$ then we can write the function as $a(k) = k^2$, for $k \geq 1$. This formula defines the sequence

$$1^2,\ 2^2,\ 3^2, \ldots$$

Notice that the variable x is replaced by the variable k. When working with sequences, the middle letters in the alphabet are usually used as variables. Moreover, it is customary to write a_k instead of $a(k)$.

The terms of a sequence can be written as

$$a_1, a_2, a_3, \ldots, a_k, \ldots$$

Example. Write the first four terms of the sequence given by

1. $a_k = 3k - 2$, $k \geq 1$.
2. $a_k = 3 + (-1)^k$, $k \geq 1$.
3. $a_k = \dfrac{(-1)^k}{2k-1}$, $k \geq 1$.

Solution.

1. $1,\ 4, 7,\ 10, \ldots$
2. $2,\ 4,\ 4,\ 2,\ 4, \ldots$
3. $-1,\ \dfrac{1}{3},\ -\dfrac{1}{5},\ \dfrac{1}{7}, \ldots$ □

An **arithmetic sequence** is a sequence of numbers in which there is a common difference d between consecutive terms.

Example. The following is an arithmetic sequence with common difference $d = 3$.

$$2,\ 5,\ 8,\ 11,\ 14, \ldots$$

Example. In this example, the common difference is a negative number, $d = -4$.

$$2,\ -2,\ -6,\ -10, \ldots$$

If we denote the first term as a, then the form of a arithmetic sequence is

$$a, a + d, a + 2d, a + 3d, \ldots.$$

A formula for an arithmetic sequence is $a_k = a + d(k - 1)$ for $k \geq 1$.

Example. Find an expression for the nth term of the arithmetic sequence.

$$2, 7, 12, 17, \ldots$$

Solution. We take the difference between successive terms, and we see that the difference is the number $d = 5$. The first term is $a = 2$. Therefore a formula for the sequence is

$$a_n = 2 + 5(k - 1) = 5k - 3, \quad k \geq 1. \qquad \square$$

Including $k \geq 1$ avoids confusion because sometimes formulas for sequences are given where the numbering does not start at the integer 1, but some at some other integer, like 4, or -2.

The formula for an arithmetic sequence will always be of the form $dk + c$, where d is the common difference and c is some number. The number c can be found easily by putting in a few values for k and then solving for c.

Example. Find a formula for the arithmetic sequence

$$4, 9, 14, 19, 24, \ldots$$

Solution. The common difference is $d = 5$. A formula for the kth term is of the form $a_k = dk + c = 5k + c$. We see that $a_1 = 4$, so $a_1 = 5(1) + c = 4$ gives $c = -1$. Thus a formula is

$$a_k = 5k - 1, \text{ for } k \geq 1.$$

We can check to see that this works for the other terms,

$$a_2 = 5(2) - 1 = 9, \quad a_3 = 5(3) - 1 = 14, \quad \text{etc.}$$ □

Exercises.

Write the first four terms of the sequence given by

1. $2a_k + 1, \quad k \geq 1$
2. $3(-2)^k, \quad k \geq 1$
3. $(-1)^k, \quad k \geq 1$
4. $(-1/3)^k, \quad k \geq 1$

Find an expression for the kth term of the arithmetic sequence, a_k.

5. $3, 7, 11, 15, 19, \ldots$
6. $-2, 1, 4, 7, 10, \ldots$
7. $3, 5, 7, 9, \ldots$
8. $3, 0, -3, -6, \ldots$
9. $2, -3, -8, -13, \ldots$

Solutions.

1. $3, 5, 7, 9, \ldots$
2. $-6, 12, -24, 48, \ldots$
3. $-1, 1, -1, 1, \ldots$

4. $-1/3, 1/9, -1/27, 1/18, \ldots$

5. $a_k = 4k - 1,\ k \geq 1$

6. $a_k = 3k - 5,\ k \geq 1$

7. $a_k = 2k + 1,\ k \geq 1$

8. $a_k = -3k + 3,\ k \geq 1$

9. $a_k = -5k + 7,\ k \geq 1$