11.2 Series

A **finite series** is a sum of numbers

$$a_1 + a_2 + a_3 + \cdots + a_n.$$

For example, the sum of numbers

$$1 + 2 + 3 + \cdots + 98 + 99 + 100$$

forms a finite series.

There is a nice story about the great mathematician Carl Friedrich Gauss as a young boy. Gauss was always finishing his work too quickly, so his teacher, to keep Gauss busy, gave him the task of adding up the counting numbers from 1 to 100. Gauss, however, surprised his teacher by completing the task very quickly.

This is how he did it. First, he wrote out the sum

$$s = 1 + 2 + 3 + \cdots + 100.$$

He then wrote out the same sum backward, and then he added the two together.

$$\begin{array}{rrcrcrcrcccrcrcr} & s & = & 1 & + & 2 & + & 3 & + & \ldots & + & 98 & + & 99 & + & 100 \\ + & s & = & 100 & + & 99 & + & 98 & + & \ldots & + & 3 & + & 2 & + & 1 \\ \hline & 2s & = & \multicolumn{13}{c}{\underbrace{101 + 101 + 101 + \ldots + 101 + 101 + 101}_{\text{100 times}}} \end{array}$$

The result was $2s$ equals 101 added 100 times. That is, $2s = 100 \cdot 101$, and therefore

$$s = \frac{(100)(101)}{2} = 5,050.$$

Don't you wish that you were as clever as the little boy Gauss!

If we add the integers from 1 to n, then by replacing n for 100 in the formula above, we get the formula

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}. \tag{1}$$

Example. Find the sum of the integers from 1 to 1000.

Solution. $1+2+3+\cdots+1000 = \dfrac{1000(1000+1)}{2} = 500 \cdot 1001 = 500,500.$ □

Arithmetic Series

If the terms of a finite series form a arithmetic sequence, then we say that the series is an **arithmetic series**. For example, the sum of the odd numbers

$$1 + 3 + 5 + \cdots + 95 + 97 + 99$$

is a finite arithmetic series because the terms $1, 3, 5, \ldots, 99$ form an arithmetic sequence.

Example. Let's find the sum of the odd numbers

$$s = 1 + 3 + 5 + \cdots + 99$$

by using the same trick that Gauss used as a little boy. We write out the sum forward, then backward, and finally we add the two together.

$$\begin{array}{rrcrcrcrcccrcrcr} & s & = & 1 & + & 3 & + & 5 & + & \ldots & + & 95 & + & 97 & + & 99 \\ + & s & = & 99 & + & 97 & + & 97 & + & \ldots & + & 5 & + & 3 & + & 1 \\ \hline & 2s & = & \multicolumn{13}{l}{\underbrace{100 + 100 + 100 + \ldots + 100 + 100 + 100}_{50 \text{ times}}} \end{array}$$

$$2s = 50(100)$$
$$s = \frac{50(100)}{2} = (50)^2 = 2500.$$

Notice that half of the integers between 1 and 100 are odd, the other half are even. Therefore, there are 50 odd numbers between 1 and 99.

In general, the sum of the first n positive odd integers is n^3.

We will now develop a formula for finding the sum of a finite arithmetic series. Let's denote the first term of an arithmetic series by a_1, the last term by a_n, and the common difference by d. We first write the series forward, then backward, we finally we add the two together.

$$\begin{array}{rl} s = & a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_1 + (n-2)d) + (a_1 + (n-1)d), \\ s = & a_n + (a_n - d) + (a_n - 2d) + \cdots + (a_n - (n-2)d) + (a_n - (n-1)d), \\ \hline 2s = & \underbrace{(a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n) + (a_1 + a_n),}_{n \text{ times}} \end{array}$$

$$2s = n(a_1 + a_n),$$
$$s = \frac{n(a_1 + a_n)}{2}.$$

Thus the sum of an arithmetic series of n terms, with first term a_1, and last term a_n is

$$s = \frac{n(a_1 + a_n)}{2}. \tag{2}$$

Example. Find the sum of the arithmetic series

$$2 - 1 - 4 - 7 - \cdots - 55.$$

Solution. The first term is $a_1 = 2$ and the last term is $a_n = -55$. It is a little more difficult to determine how many terms there are. The common difference is $d = -3$, so a formula for the kth term is of the form $a_k = -3k+c$. By putting in $k = 1$, we see that $a_k = -3k + 5$. We have that $a_n = -3n + 5$, and $a_n = -55$, so it follows that $n = 20$. We put this all into formula (2) and get

$$s = \frac{n(a_1 + a_n)}{2} = \frac{20(2 + (-55))}{2} = -530. \qquad \square$$

Sigma Notation

A convenient way to write the sum of a finite series of numbers is to use the Greek letter Σ, upper case sigma.

Returning to the previous example, the sum of the odd numbers,

$$1 + 3 + 5 + \cdots + 99,$$

form an arithmetic series, with first term $a = 1$, common difference $d = 2$, and $n = 50$ terms. A formula for the kth term is $a_k = a + d(k-1) = 1 + 2(k-1) = 2k - 1$. We can write the series as

$$1 + 3 + 5 + \cdots + 99 = \sum_{k=1}^{50} (2k - 1).$$

Below the sigma is the starting number for the variable k; in this case, k starts at 1. On the top of the sigma, we write the ending value for the number

k; in this case, k ends at 50. The variables commonly used in summation notation are i, j, k, l, m, and n. When we write a finite series this way, it is said to be written using sigma notation or summation notation. The variables in summation notation only take integer values.

Example. Expand the series that is given in sigma notation. Do not evaluate the sum.

$$s = \sum_{k=1}^{4} (2k^2 + 1)$$

Solution.

$$s = \sum_{k=1}^{4} (2k^2 + 1) = (2(1)^2 + 1) + (2(2)^2 + 1) + (2(3)^2 + 1) + (2(4)^2 + 1).$$

□

Example. Write the series using summation notation,

$$2 + 5 + 8 + 11 + 14 + 17 + 20.$$

Solution. The sequence

$$2, 5, 8, 11, 14, 17, 20$$

is an arithmetic sequence with common difference $d = 3$. Therefore a formula for the kth term is of the form $a_k = 3k + c$. By putting in $k = 1$, we see that $c = -1$, so we get $a_k = 3k - 1$ as a formula for the kth term. We also see that the first term corresponds to $k = 1$, and the last term corresponds to $k = 7$. Putting this all together gives

$$\sum_{k=1}^{7} (3k - 1).$$

□

Geometric Series

A **geometric series** is a series of numbers in which each term after the first is obtained from the one before it by multiplying by a fixed number, r, called the **common ratio**.

An example of a finite geometric series is

$$1 + 2 + 4 + 8 + 16 + 32.$$

The first term is $a = 1$ and the common ratio is the number $r = 2$.

A finite geometric series with n terms can be written as

$$a + ar + ar^2 + \cdots + ar^{n-1}.$$

Using sigma notation and using the variable k starting at 1 and going to n, we write a geometric series as

$$a + ar + ar^2 + \cdots + ar^{n-1} = \sum_{k=1}^{n} ar^{k-1}.$$

Let's find a formula for the sum. Denote the sum of the series as s and write it out, and then multiply the series by r.

$$\begin{aligned} s &= a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}, \\ rs &= r(a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}) \\ &= ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \end{aligned}$$

Now subtract and cancel terms. We get

$$\begin{aligned} s - rs &= a - ar^n. \\ s(1-r) &= a(1-r^n) \\ s &= \frac{a(1-r^n)}{1-r}. \end{aligned}$$

Example. Find the sum of the geometric series given by

$$1 + 2 + 4 + 8 + 16 + 32 + 64 + \cdots + 2^{99}.$$

Solution. The series can be written as

$$2^{1-1} + 2^{2-1} + 2^{3-1} + \cdots + 2^{100-1}.$$

The first term is $a = 2$ and the common ratio is $r = 2$. There are $n = 100$ terms. The sum is then given by

$$s = \frac{a(1-r^n)}{1-r} = \frac{1(1-2^{100})}{1-2} = 2^{100} - 1. \qquad \square$$

If the common ratio, r, is a number such that

$$-1 < r < 1.$$

Then for very large n, the number r^n is very small. For example, if $r = 1/2$ and $n = 100$, then

$$r^n = \left(\frac{1}{2}\right)^{100} = \frac{1}{2^{100}}.$$

This is a very small number.

We say that as n approaches infinity, r^n approaches 0, provided that $-1 < r < 1$. The expression r^n appears in the formula for the sum of a geometric series. As n approaches infinity,

$$s = \frac{a(1-r^n)}{1-r} \quad \text{approaches} \quad \frac{a(1-0)}{1-r} = \frac{1}{1-r}.$$

If we let n approach infinity, then the sum is called an infinite geometric series and is written as

$$a + ar + ar^2 + \ldots$$

Using summation notation, we write

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ldots$$

Thus the sum of an infinite geometric series is

$$a + ar + ar^2 + \ldots \frac{a}{1-r}, \text{ when } -1 < r < 1. \qquad (3)$$

Example. Any infinite repeating decimal can be represented as a rational number. For instance,

$$0.333\cdots = \frac{1}{3}.$$

To show why this is true, we can write this repeating decimal as a geometric series. We have

$$\begin{aligned} 0.333\ldots &= 0.3 + 0.03 + 0.003 + \ldots \\ &= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots \end{aligned}$$

The first term is $a = 3/10$, and the common ration is $r = 1/10$. Thus, the sum is

$$s = \frac{a}{1-r} = \frac{3/10}{1-1/10} = \frac{3/10}{9/10} = \frac{1}{3}.$$

Exercises.

Find the sum of the arithmetic series using formula (2).

1. $1 + 2 + 3 + \cdots + 500$
2. $2 + 4 + 6 + 8 + \cdots + 100.$
3. $1 + 4 + 7 + 10 + 13 + \cdots + (3k-2) + \cdots + 298.$
4. $2 - 2 - 6 - 10 - 14 - \cdots - 74.$
5. $11 + 12 + 13 + 14 + \cdots + 80.$

Write the sum in expanded form. Do not evaluate the sum.

6. $\displaystyle\sum_{i=1}^{5} \sqrt{i}$
7. $\displaystyle\sum_{i=1}^{6} 3^i$
8. $\displaystyle\sum_{k=0}^{4} \frac{2k-1}{2k+1}$

9. $\sum_{k=1}^{8} k^3$

Write the sum using sigma notation.

10. $1+3+5+7+9+11+13+15$

11. $1+4+9+16+25+36$

12. $2+4+6+\cdots+2n$

13. $1+2+4+8+16+32$

14. $x+x^2+x^3+\cdots+x^{10}$

Using formula (3) for the sum of an infinite geometric series, find the sum.

15. $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$

16. $4-2+1-\frac{1}{2}+\frac{1}{4}-\ldots$

17. $6-2+\frac{2}{3}-\ldots$

Express as a fraction.

18. $0.\overline{6}=0.666\ldots$

19. $0.\overline{23}=0.232323\ldots$

20. $5.72\overline{5}=5.72555\ldots$

Solutions.

Find the sum of the arithmetic series using formula (2).

1. $s=1+2+3+\cdots+500$, $n=500$, using formula (1), $s=\frac{n(n+1)}{2}=\frac{500(500+1)}{2}=250\cdot 501=125,250$.

2. $s=2+4+6+8+\cdots+100$. $a_k=2k$, $a_n=2n=100$, $n=50$. $a_1=2$, $a_{50}=100$, us-

ing formula (2), $s = \frac{n(a_1+a_n)}{2} = \frac{50(2+100)}{2} = 50 \cdot 51 = 2550$.

3. $s = 1+4+7+10+13+\cdots+(3k-2)+\cdots+298$. $a_k = 3k-2$, $a_n = 3n-2 = 298$, $n = 100$, $a_1 = 1$, $a_{100} = 298$, $s = \frac{n(a_1+a_n)}{2} = \frac{100(1+298)}{2} = 50 \cdot 299 = 14950$.

4. $s = 2-2-6-10-14-\cdots-74$. $a = 2$, $d = -4$, $a_k = 2 + (-4)(k-1) = -4k+6$. $a_n = -4n+6 = -74$, $n = 20$, $a_1 = 2$, $a_20 = -74$, $s = \frac{n(a_1+a_n)}{2} = \frac{20(2+(-74))}{2} = 10 \cdot (-72) = -720$.

5. $s = 11+12+13+14+\cdots+80$. $a_k = k+10$, $1 \le k \le 70$, $n = 70$, $a_1 = 11$, $a_70 = 80$, $s = \frac{n(a_1+a_n)}{2} = \frac{70(11+80)}{2} = 35 \cdot 91 = 3185$.

Write the sum in expanded form. Do not evaluate the sum.

6. $\sum_{i=1}^{5} \sqrt{i} = \sqrt{1}+\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{5}$

7. $\sum_{i=1}^{6} 3^i = 3+3^2+3^3+3^4+3^5+3^6$

8. $\sum_{k=0}^{4} \frac{2k-1}{2k+1} = -1+1/3+3/5+5/7+7/9$

9. $\sum_{k=1}^{8} k^3 = 1^3+2^3+3^3+4^3+5^3+6^3+7^3+8^3$

Write the sum using sigma notation.

10. $1+3+5+7+9+11+13+15 = \sum_{k=1}^{8}(2k+1)$

11. $1+4+9+16+25+36 = \sum_{k=1}^{6} k^2$

12. $2+4+6+\cdots+2n = \sum_{k=1}^{n} 2i$

13. $1+2+4+8+16+32 = \sum_{k=0}^{5} 2^k$

14. $x+x^2+x^3+\cdots+x^{10} = \sum_{k=1}^{10} x^k$

Using formula (3) for the sum of an infinite geometric series, find the sum.

15. $s = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$, $a = 1$, $r = 1/2$, $s = \frac{a}{1-r} = \frac{1}{1-1/2} = 2$.

16. $s = 4-2+1-\frac{1}{2}+\frac{1}{4}-\ldots$. $a = 4$, $r = -1/2$, $s = \frac{a}{1-r} = \frac{4}{1-(-1/2)} = 8/3$

17. $s = 6-2+\frac{2}{3}-\ldots$, $a = 6$, $r = -1/3$, $s = \frac{a}{1-r} = \frac{6}{1-(-1/3)} = 9/2$.

Express as a fraction.

18. $0.\overline{6} = 0.666\cdots = 0.6 + 0.06 + 0.006 + \ldots$, $a = 0.6$, $r = 0.1$, $s = \frac{a}{1-r} = \frac{0.6}{1-0.1} = 2/3$

19. $0.\overline{23} = 0.232323\cdots = 0.23 + 0.0023 + 0.000023 + \ldots$, $a = 0.23$, $r = 0.01$, $s = \frac{a}{1-r} = \frac{0.23}{1-0.01} = \frac{.23}{0.99} = \frac{23}{99}$.

20. $5.72\overline{5} = 5.72555\cdots = 5.72 + 0.005 + 0.0005 + 0.00005 + \ldots$, $a = .005$, $r = 0.1$, $s = 5.72 + \frac{a}{1-r} = \frac{572}{100} + \frac{0.005}{1-0.1} = \frac{572}{100} + \frac{0.005}{0.9} = \frac{572}{100} + \frac{5}{900} = \frac{572 \cdot 9 + 5}{900} = \frac{5153}{900}$ (irreducible).