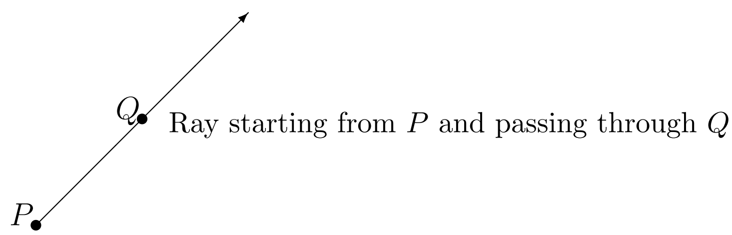
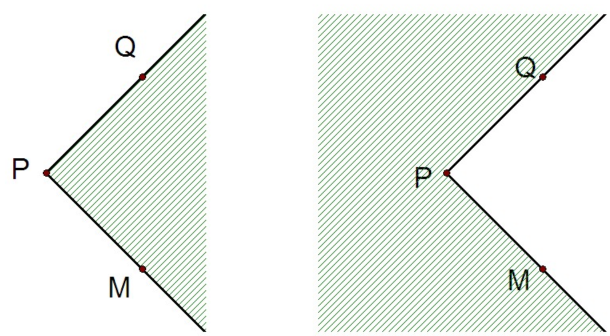
5.1 Angles and Their Measurements

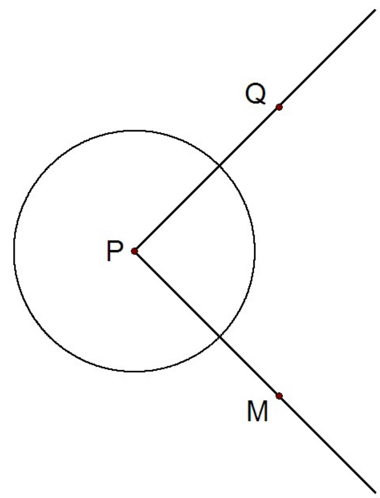
A **ray** is a half line. It consists of all the points on a line through a point P that lie to one side of P. The ray starting from P and passing through another point Q will be denoted R_{PQ}. We call P the **vertex** of the ray.



Let's consider two rays R_{PQ} and R_{PM} starting from the same point P. These rays separate the plane into two regions. Each of these regions will be called an **angle** determined by the rays.

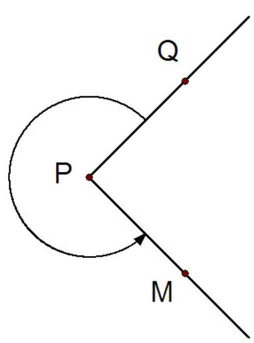


Let R_{PQ} and R_{PM} be rays with vertex P. If C is a circle centered at P, then the two rays separate the circle into two arcs. Each arc lies within one of the angles, and therefore to characterize an angle it suffices to draw the corresponding arc.

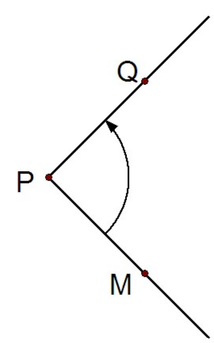


Again consider two rays R_{PQ} and R_{PM} starting from the same point P. The rays determine two different angles. One angle contains the arc going in the **counterclockwise** direction from ray R_{PQ} to ray R_{PM}. We will denote this angle as $\angle QPM$.

The other angle determined by rays R_{PQ} and R_{PM} contains the arc going in the counterclockwise direction from ray R_{PM} to ray R_{PQ}. We will denote this angle as $\angle MPQ$.

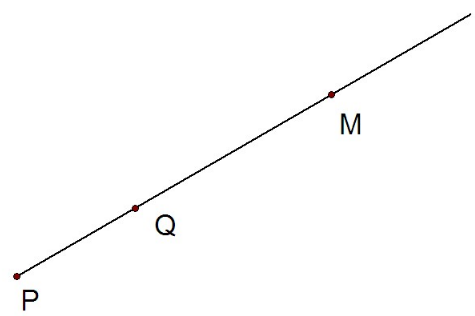


$\angle QPM$



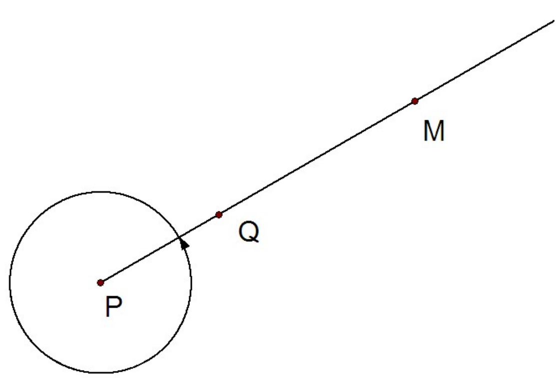
$\angle MPQ$

Example If P, Q, and M lie on the same straight line, and Q, M lie on the same ray starting at P, then the angle $\angle QPM$ looks like this:



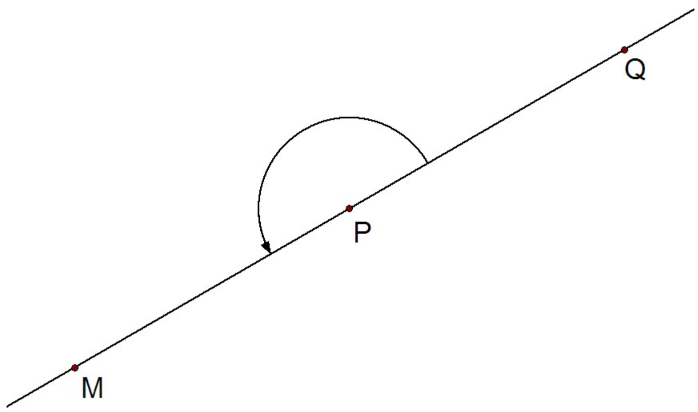
In this case, the arc of a circle between the two rays is just a point, and we say that the angle $\angle PQM$ is the **zero angle**.

The other angle determined by these two rays that coincide is the whole plane, and is called the **full** angle.

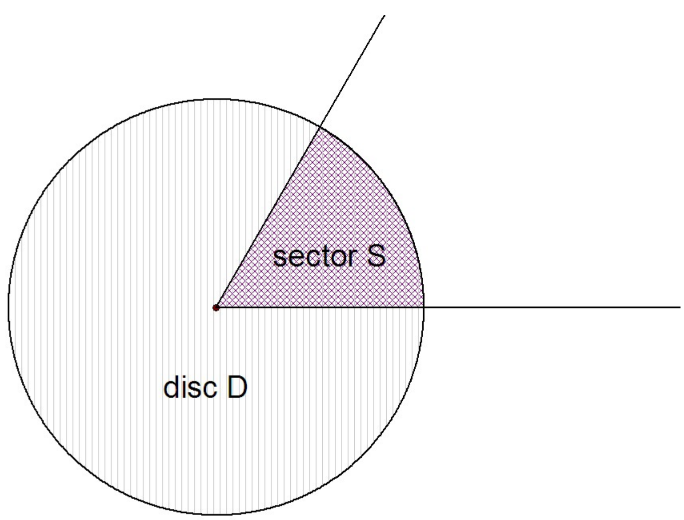


Example Suppose that Q, P, and M lie on the same straight line but that Q and M do not lie on the same ray. That is, Q and M lie on opposite sides of P on the line. Our angle $\angle QPM$ looks like this:

In this case, we say that the angle $\angle QPM$ is a **straight angle**.



Given an angle A with vertex P, let D be a disc centered at P. That part of the angle which also lies in the disc is called the **sector** of the disc determined by the angle.



We can measure an angle using **degrees**. A full angle has 360 degrees. Let A be an angle centered at P and let S be the sector determined by A in the disc D centered at P. Let θ be a number between 0 and 360. We shall say

angle A has θ degrees

to mean that

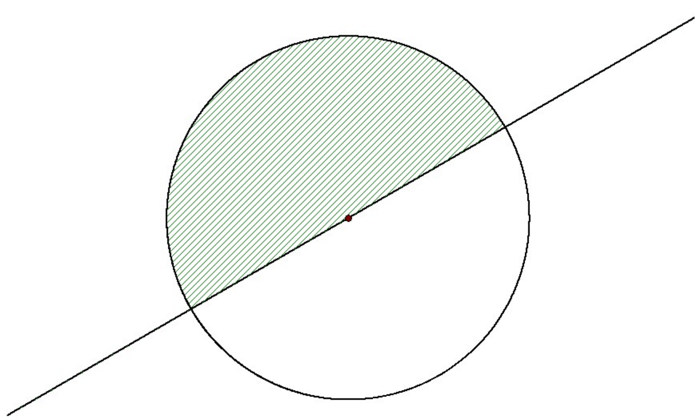
$$\frac{\text{area of sector } S}{\text{area of disc } D} = \frac{\theta}{360}$$

Thus

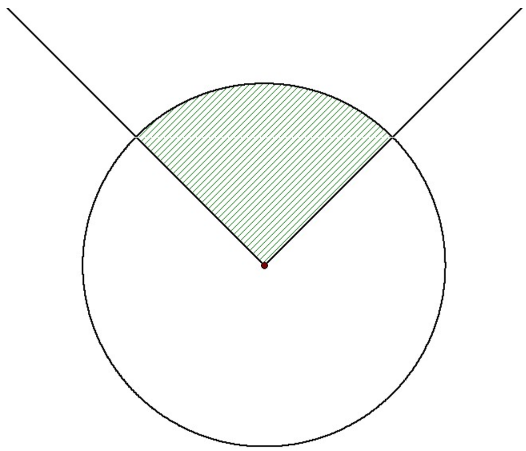
$$\theta = 360 \cdot \frac{\text{area of sector } S}{\text{area of disc } D}$$

In computing the number of degrees in an angle, we do not have to determine the areas of the sector S or the sector D. We only need to determine the ratio between the two.

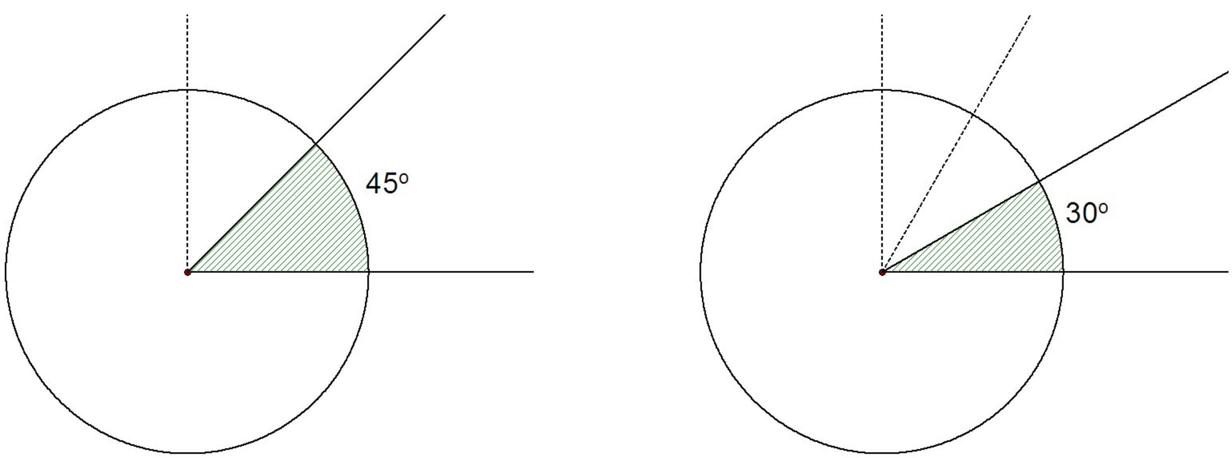
Example The straight angle has 180 degrees because it separates the disc into two sectors of equal area.



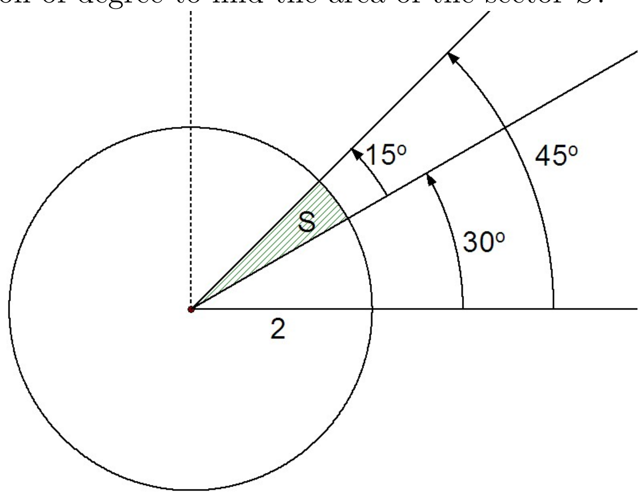
Example An angle whose measure is half of the straight angle is called a **right angle**, and has 90 degrees.



Example The angle with 45 degrees is half of a right angle. The angle with 30 is one third of a right angle. The angle with measure 60 degrees is two thirds of a right angle.



Example We can find the area of the sector S lying between angles of 30° and 45°, and inside the circle of radius 2. Let us assume that the area of the disc of radius r is given by $A = \pi r^2$. The area of the disc of radius 2 is then $\pi(2)^2 = 4\pi$. The sector S has 15° because $15 = 45 - 30$. We can use the definition of degree to find the area of the sector S.



$$\frac{\text{Area of sector } S}{\text{area of disk } D} = \frac{\theta}{360}$$

$$\frac{\text{Area of sector} S}{4\pi} = \frac{15}{360}$$

$$\text{area of } S = \frac{15}{360} \cdot 4\pi = \frac{\pi}{6}.$$

Radian Measure

Let π be the area of a disc of radius 1. Let A be an angle with vertex P and let S be the sector determined by A in the disc D of radius 1 centered at P. Let x be a number between 0 and 2π. We say that

$$A \textbf{ has } \theta \textbf{ radians}$$

to mean that

$$\frac{\text{Area of sector } S}{\text{Area of disc } D} = \frac{\theta}{2\pi}.$$

We can find a formula for the area of the sector S if θ is given in radians.

$$\frac{\text{Area of sector } S}{\pi r^2} = \frac{\theta}{2\pi}.$$

$$\text{Area of sector } S = \pi r^2 \cdot \frac{\theta}{2\pi}.$$

$$\text{Area of sector } S = \frac{r^2\theta}{2}.$$

Of course, degrees are related to radians. For instance:

$$\begin{aligned} 360^\circ &= 2\pi \text{ radians} \\ 180^\circ &= \pi \text{ radians} \\ 90^\circ &= \pi/2 \text{ radians} \\ 60^\circ &= \pi/3 \text{ radians} \\ 45^\circ &= \pi/4 \text{ radians} \\ 30^\circ &= \pi/6 \text{ radians} \end{aligned}$$

In general,

$$\theta \text{ degrees } = \frac{\pi}{180}\theta \text{ radians.}$$

Arc Length

The circumference of a circle of radius r is $2\pi r$. The angle A determines an arc s on this circle, and measure of the angle is proportional to the arc length. If θ is the measure of an angle A in radians, then we have

$$\frac{\text{length of arc } s}{\text{total length of circle}} = \frac{\theta}{2\pi}.$$

that is,

$$\frac{\text{length of arc } s}{2\pi r} = \frac{\theta}{2\pi}.$$

$$\text{length of arc } s = 2\pi r \frac{\theta}{2\pi}.$$

$$\text{length of arc } s = r\theta.$$

If the radius $r = 1$, then the radian measure of an angle equals the length of the arc.

$$\frac{\text{Area of sector } S}{\text{Area of disc } D} = \frac{\text{length of arc } s}{\text{total length of circle}} = \frac{\theta}{2\pi}.$$

Example Find the length of the arc s of a circle with radius 2 intercepted by an angle of $\pi/4$ radians.

Solution

$$\frac{\text{length of arc } s}{2\pi r} = \frac{\theta}{2\pi}.$$

$$\frac{\text{length of arc } s}{2\pi(2)} = \frac{\pi/4}{2\pi}.$$

$$\text{length of arc } s = 4\pi \cdot \frac{\pi}{4} \cdot \frac{1}{2\pi} = \frac{\pi}{2} \qquad \square$$

For convenience of language, we shall sometimes speak, incorrectly but usefully, of an angle of θ radians even if θ does not lie between 0 and 2π. To do this, we write

$$\theta = 2n\pi + w,$$

with a number w such that

$$0 \leq w \leq 2\pi.$$

Then, by an angle of θ radians, we mean the angle of w radians.

Also for convenience of language, if θ is negative, say

$$\theta = -\alpha$$

where α is positive, we shall speak of angle of θ radians to mean an angle of α radians in the clockwise direction.