6.4 Double-Angle and Half Angle Identities

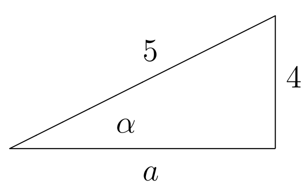
$$\sin 2\theta = 2\sin\theta\cos\theta, \quad \cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\cos 2\theta = 1 - 2\sin^2\theta, \quad \cos 2\theta = 2\cos^2\theta - 1$$
$$\sin^2\theta = \frac{1-\cos 2\theta}{2}, \quad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$
$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}, \quad \sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$$

1. Find $\cos 2\alpha$ given that $\sin\alpha = -4/5$ and α lies in quadrant III.

   Solution

   $$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

   Find $\cos\alpha$, given that α lies in quadrant III and $\sin\alpha = -\frac{4}{5}$.

   $\sin\alpha = \frac{\text{opp}}{\text{hyp}} = -\frac{4}{5}$

   Use the Pythagorean Theorem to find b.
   $4^2 + b^2 = 5^2$
   $16 + b^2 = 25$
   $b^2 = 9$
   $b = 3$
   $\cos\alpha = \dfrac{\text{adj}}{\text{hyp}} = -\dfrac{3}{5}$ ←Negative because α lies in QII.

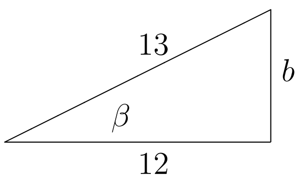
   $$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \left(-\frac{3}{5}\right)^2 - \left(-\frac{4}{5}\right)^2 = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}$$

2. Find $\sin 2\beta$ given that $\cos\beta = 12/13$ and β lies in quadrant IV.

SOLUTION

$$\sin 2\beta = 2\sin\beta\cos\beta$$

Find $\sin\beta$ given that β lies in quadrant IV and $\cos\beta = \frac{12}{13}$.



$\cos\beta = \dfrac{\text{adj}}{\text{hyp}} = \frac{12}{13}$

Use the Pythagorean Theorem to find b.
$12^2 + b^2 = 13^2$
$144 + b^2 = 169$
$b^2 = 25$
$b = 5$
$\sin\beta = \dfrac{\text{opp}}{\text{hyp}} = -\dfrac{5}{13}$ ←Negative because β lies in QIV.

$$\sin 2\beta = 2\sin\beta\cos\beta = 2\left(-\frac{5}{13}\right)\left(\frac{12}{13}\right) = -\frac{120}{169}$$

3. Find $\sin\frac{\theta}{2}$ given that $\cos\theta = -1/4$ and $\pi < \theta < \frac{3\pi}{2}$

SOLUTION

$$\begin{aligned}\sin\frac{\theta}{2} &= \pm\sqrt{\frac{1-\cos\theta}{2}} \\ &= \pm\sqrt{\frac{1}{2}\left(1-\frac{-1}{4}\right)}\end{aligned}$$

$$
\begin{aligned}
&= \pm\sqrt{\frac{1}{2}\left(\frac{5}{4}\right)} \\
&= \pm\sqrt{\frac{5}{8}} \\
&= \pm\frac{\sqrt{5}}{\sqrt{8}} = \pm\frac{\sqrt{5}}{2\sqrt{2}} = \pm\frac{\sqrt{5}\sqrt{2}}{4} = \pm\frac{\sqrt{10}}{4}
\end{aligned}
$$

Because $\pi < \theta < \frac{3\pi}{2}$, it follows that $\frac{\pi}{2} < \frac{\theta}{2} < \frac{3\pi}{4}$. Therefore, $\theta/2$ lies in quadrant II and $\sin\frac{\theta}{2}$ is positive. Answer: $\sin\frac{\theta}{2} = \frac{\sqrt{10}}{4}$.

4. Find $\cos\frac{\theta}{2}$ given that $\sin\theta = -12/13$ and $\frac{7\pi}{4} < \frac{\theta}{2} < 2\pi$

   SOLUTION

   $\frac{7\pi}{4} < \frac{\theta}{2} < 2\pi$ implies that $\frac{7\pi}{2} < \theta < 4\pi$, which means that θ lies in quadrant IV.

$$
\begin{aligned}
\cos^2\theta &= 1 - \sin^2\theta \\
&= 1 - \left(\frac{-12}{13}\right)^2 \\
&= 1 - \frac{144}{169} = \frac{25}{169} \\
\cos\theta &= \sqrt{\frac{25}{169}} = \frac{5}{13} \quad \leftarrow \text{positive because } \theta \text{ lies in QIV} \\
\cos\frac{\theta}{2} &= \pm\sqrt{\frac{1+\cos\theta}{2}} \\
&= \pm\sqrt{\frac{1}{2}\left(1 + \frac{5}{13}\right)} \\
&= \pm\sqrt{\frac{1}{2}\left(\frac{18}{13}\right)} = \pm\sqrt{\frac{9}{13}} = \pm\frac{3}{\sqrt{13}} = \pm\frac{3\sqrt{13}}{13} \\
\cos\frac{\theta}{2} &= \frac{3\sqrt{13}}{13} \quad \leftarrow \text{positive because } \frac{\theta}{2} \text{ lies in QIV}
\end{aligned}
$$

5. Find $\sin\theta$ given that $\cos(2\theta) = 3/5$ and $0 < 2\theta < \frac{\pi}{2}$

   SOLUTION

$$\begin{aligned}
\sin\frac{A}{2} &= \pm\sqrt{\frac{1-\cos A}{2}} \\
\sin\theta &= \pm\sqrt{\frac{1-\cos 2\theta}{2}} \\
&= \pm\sqrt{\frac{1}{2}\left(1-\frac{3}{5}\right)} \\
&= \pm\sqrt{\left(\frac{1}{2}\right)\left(\frac{2}{5}\right)} = \pm\frac{1}{\sqrt{5}} == \pm\frac{\sqrt{5}}{5}
\end{aligned}$$

   $0 < 2\theta < \frac{\pi}{2}$ implies that $0 < \theta < \frac{\pi}{4}$. Therefore, $\sin\theta$ is positive.

   Answer: $\sin\theta = \frac{\sqrt{5}}{5}$

6. Find $\cos\theta$ given that $\sin(2\theta) = 5/13$ and $0 < \theta < \frac{\pi}{4}$

   If $\sin(2\theta) = 5/13$ and $0 < \theta < \frac{\pi}{4}$, then $0 < 2\theta < \frac{\pi}{2}$ and

$$\begin{aligned}
\cos^2 2\theta &= 1-\sin^2 2\theta \\
&= 1-\left(\frac{5}{13}\right)^2 \\
&= 1-\frac{25}{169} = \frac{144}{169} \\
\cos 2\theta &= \frac{12}{13} \leftarrow \text{positive because } 2\theta \text{ in QI}
\end{aligned}$$

$$\begin{aligned}
\cos\frac{A}{2} &= \pm\sqrt{\frac{1+\cos A}{2}} \\
\cos\theta &= \pm\sqrt{\frac{1+\cos 2\theta}{2}} \\
&= \pm\sqrt{\frac{1}{2}\left(1+\frac{12}{13}\right)}
\end{aligned}$$

$$\begin{aligned}
&= \pm\sqrt{\frac{1}{2}\left(1+\frac{12}{13}\right)} \\
&= \pm\sqrt{\frac{1}{2}\left(\frac{25}{13}\right)} \\
&= \pm\sqrt{\frac{25}{26}} \leftarrow \text{positive because } \theta \text{ in QI} \\
&= \frac{5}{\sqrt{26}} = \frac{5\sqrt{26}}{26}
\end{aligned}$$

7. Write $\cos^4\theta$ in terms of $\cos\theta$.

$$\begin{aligned}
\cos^4\theta &= \left(\cos^2\theta\right)^2 \\
&= \left(\frac{1+\cos 2\theta}{2}\right)^2 \\
&= \frac{1}{4}\left(1+2\cos 2\theta+\cos^2 2\theta\right) \\
&= \frac{1}{4}\left(1+2\cos 2\theta+\frac{1}{2}(1+\cos 4\theta)\right) \\
&= \frac{1}{4}+\frac{1}{2}\cos 2\theta+\frac{1}{8}+\frac{1}{8}\cos 4\theta) \\
&= \frac{3}{8}+\frac{1}{2}\cos 2\theta+\frac{1}{8}\cos 4\theta
\end{aligned}$$