6.4 Double-Angle Identities

Here are some more trigonometric identities. You will have to know some of these identities is calculus 2.

Double-Angle Identities

(1) $\sin 2\theta = 2\sin\theta\cos\theta$

(2) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

(3) $\cos 2\theta = 1 - 2\sin^2\theta$

(4) $\cos 2\theta = 2\cos^2\theta - 1$

(5) $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

PROOF OF (1)

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

Let $\alpha = \theta$ and $\beta = \theta$

$$\begin{aligned}\sin(\theta+\theta) &= \sin\theta\cos\theta + \cos\theta\sin\theta\\ \sin(2\theta) &= 2\sin\theta\cos\theta \quad \square\end{aligned}$$

PROOF OF (2)

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Let $\alpha = \theta$ and $\beta = \theta$.

$$\begin{aligned}\cos(\theta+\theta) &= \cos\theta\cos\theta - \sin\theta\sin\theta\\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta \quad \square\end{aligned}$$

PROOF OF (3)

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta\\ \cos(2\theta) &= (1-\sin^2\theta) - \sin^2\theta\\ \cos(2\theta) &= 1 - 2\sin^2\theta \quad \square\end{aligned}$$

PROOF OF (4)

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ \cos(2\theta) &= \cos^2\theta - (1-\cos^2\theta) \\ \cos(2\theta) &= 2\cos^2\theta - 1 \quad \square\end{aligned}$$

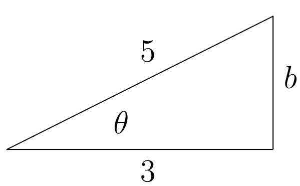
PROOF OF (5)

$$\begin{aligned}\tan(2\theta) &= \frac{\sin 2\theta}{\cos 2\theta} \\ &= \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} \\ &= \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} \cdot \left(\frac{1/\cos^2\theta}{1/\sin^2\theta}\right) \\ &= \frac{2\sin\theta/\cos\theta}{1-\sin^2\theta/\cos^2\theta} \\ &= \frac{2\tan\theta}{1-\tan^2\theta} \quad \square\end{aligned}$$

Example Given $\cos\theta = 3/5$ and θ lies in quadrant III, find $\sin 2\theta$, $\cos 2\theta$.

SOLUTION

First find $\sin\theta$.



$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{3}{5}$

Use the Pythagorean Theorem to find b.
$3^2 + b^2 = 5^2$
$9 + b^2 = 25$
$b^2 = 16$

$b = 4$
$\sin\theta = \dfrac{\text{opp}}{\text{hyp}} = -\dfrac{4}{5}$ ←Negative because θ lies in QIV.

$$\begin{aligned} \sin 2\theta &= 2\sin\theta\cos\theta = 2\left(-\frac{4}{5}\right)\left(\frac{3}{5}\right) = -\frac{24}{25} \\ \cos 2\theta &= \cos^2\theta - \sin^2\theta = \left(\frac{3}{5}\right)^2 - \left(-\frac{4}{5}\right)^2 = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25} \quad \square \end{aligned}$$

The following identities follow from the double-angle identities.

$$(1) \quad \sin^2\theta = \frac{1-\cos 2\theta}{2}, \qquad (2) \quad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

Proof of (1)

$$\begin{aligned} \cos 2\theta &= 1 - 2\sin^2\theta \\ 2\sin^2\theta &= 1 - \cos 2\theta \\ \sin^2\theta &= \frac{1-\cos 2\theta}{2} \quad \square \end{aligned}$$

Proof of (2)

$$\begin{aligned} \cos 2\theta &= 2\cos^2\theta - 1 \\ 2\cos^2\theta &= 1 + \cos 2\theta \\ \cos^2\theta &= \frac{1+\cos 2\theta}{2} \quad \square \end{aligned}$$

Example Write $\sin^4 x$ in terms of $\cos x$.

SOLUTION

$$\begin{aligned}
\sin^4\theta &= \left(\sin^2\theta\right)^2 \\
&= \left(\frac{1-\cos 2\theta}{2}\right)^2 \\
&= \frac{1}{4}\left(1 - 2\cos 2\theta + \cos^2 2\theta\right) \\
&= \frac{1}{4}\left(1 - 2\cos 2\theta + \frac{1}{2}(1+\cos 4\theta)\right) \\
&= \frac{1}{4} - \frac{1}{2}\cos 2\theta + \frac{1}{8} + \frac{1}{8}\cos 4\theta) \\
&= \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta \quad \square
\end{aligned}$$

Half-Angle Identities

(1) $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$

(2) $\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$

(3) $\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$

(4) $\tan\frac{\theta}{2} = \frac{\sin A}{1+\cos\theta}$

(5) $\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}$

PROOF OF (1)

$$\begin{aligned}
\cos^2\alpha &= \frac{1+\cos 2\alpha}{2} \\
\cos\alpha &= \pm\sqrt{\frac{1+\cos 2\alpha}{2}}
\end{aligned}$$

$$\text{Let } \alpha = \frac{\theta}{2}$$

$$\begin{aligned}
\cos\frac{\theta}{2} &= \pm\sqrt{\frac{1+\cos 2\left(\frac{\theta}{2}\right)}{2}} \\
\cos\frac{\theta}{2} &= \pm\sqrt{\frac{1+\cos\theta}{2}} \quad \square
\end{aligned}$$

The proof of (2) is similar. The proofs of (3), (4), and (5) involve using the identity $\tan\theta/2 = \frac{\sin\theta/2}{\cos\theta/2}$ and using (1) and (2).

Example Find the exact value of $\cos 15°$ using the half-angle identity.

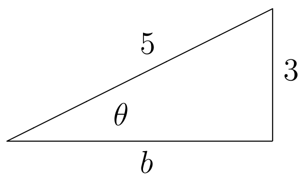
Solution The value of $\cos 15°$ will be positive because $15°$ is an acute angle.

$$\begin{aligned}
\cos 15° &= \cos\frac{30°}{2} \\
&= \sqrt{\frac{1+\cos 30°}{2}} \\
&= \sqrt{\frac{1}{2}\left(1+\frac{\sqrt{3}}{2}\right)} \\
&= \sqrt{\frac{1}{2}\left(\frac{2}{2}+\frac{\sqrt{3}}{2}\right)} \\
&= \sqrt{\frac{1}{2}\left(\frac{2+\sqrt{3}}{2}\right)} \\
&= \sqrt{\frac{2+\sqrt{3}}{4}} \\
&= \frac{\sqrt{2+\sqrt{3}}}{2} \quad \square
\end{aligned}$$

Example Given $\sin\theta = 3/5$, with θ in quadrant II, find $\sin\theta/2$.

SOLUTION

First find $\cos\theta$.



$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{3}{5}$

We could use the Pythagorean Theorem to find b, but if we remember that $3, 4, 5$ is a Pythagorean Triplet, and then we get $b = 4$.

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = -\frac{4}{5} \quad \leftarrow\text{Negative because } \theta \text{ lies in QII.}$$

$$\begin{aligned}
\sin\frac{\theta}{2} &= \pm\sqrt{\frac{1-\cos\theta}{2}} \\
&= \pm\sqrt{\frac{1-\left(-\frac{4}{5}\right)}{2}} \\
&= \pm\sqrt{\frac{1}{2}\left(\frac{5}{5}+\frac{4}{5}\right)} \\
&= \pm\sqrt{\frac{1}{2}\left(\frac{9}{5}\right)} \\
&= \pm\sqrt{\frac{9}{10}} = \pm\frac{3}{\sqrt{10}} = \pm\frac{3\sqrt{10}}{10}
\end{aligned}$$

If θ lies in quadrant II, then $90^\circ < \theta < 180^\circ$ and therefore $45^\circ < \theta/2 < 90^\circ$. We see then that $\theta/2$ lies in quadrant I, so that the value of $\sin\theta/2$ will be positive.

ANSWER: $\sin\theta/2 = \frac{3\sqrt{10}}{10}$ □

6.5 Product and Sum Identities (optional section)

Product-to-Sum Identities

$$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

Example Using a Product-to-Sum Identity. Write $4\cos 75° \sin 25°$ as the sum or difference of two functions.

Sum-to-Product Identities

Sum-to-Product Identities

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

Example Using a Sum-to-Product Identity. Write $\sin 2\theta - \sin 4\theta$ as a product of two functions.