5.5 Inverse Trigonometric Functions

The Inverse Sine Function

Suppose you are asked to give the value of one angle θ that satisfies the equation

$$\sin\theta = 1/2.$$

There are an infinite number of possibilities, but if you had to pick one, it would be reasonable to look for an angle in quadrant I, because that is everyone's favorite quadrant.

Let's rephrase the question. Solve for θ where $\sin\theta = 1/2$, and $0 \le \theta \le \pi/2$. The answer is $\theta = \pi/6$.

Now suppose that you are asked to give the value of one angle θ that satisfies

$$\sin\theta = -1/2.$$

Because quadrant I is your favorite quadrant, you would like to look there for an answer. However, the values of sine are all positive in quadrant I. Therefore, you instead go to the quadrant directly next to quadrant I where sine takes negative values. You look in quadrant IV.

Let's rephrase the question. Solve for θ where $\sin\theta = -1/2$, and $-\pi/2 \le \theta \le 0$. The answer is $\theta = -\pi/6$.

Inverse Sine Function
$\theta = \sin^{-1} x$ or $\theta = \arcsin x$ means that $x = \sin\theta$ and $-\pi/2 \le \theta \le \pi/2$.

The inverse sine function returns only one value because it is a function. That's what functions do. They only return one value.

Example Try it on your calculator. Set the calculator to degrees and punch in $\sin^{-1}(1/2)$. What do you get? Now punch in $\sin^{-1}(-1/2)$. What do you get?

Example Evaluate each expression.

1. $\arcsin(\sqrt{3}/2)$

2. $\sin^{-1}(-\sqrt{2}/2)$
3. $\sin^{-1}(-1)$
4. $\sin^{-1} 0$

The Inverse Cosine Function

Now let's define the inverse cosine function. If you had to give one answer for θ where $\cos\theta = 1/2$, then it would be reasonable to look for your one answer in quadrant I.

But if you had to pick one answer for $\cos\theta = -1/2$, which quadrant would you look in? Because quadrant I is your favorite quadrant, you will go to the quadrant closest to quadrant I where cosine is negative. That's quadrant II.

Inverse Cosine Function
$\theta = \cos^{-1} x$ or $\theta = \arccos x$ means that $x = \cos\theta$ and $0 \leq \theta \leq \pi$.

Example Evaluate each expression.

1. $\arccos \sqrt{3}/2$
2. $\cos^{-1}(-1/2)$
3. $\cos^{-1}(-\sqrt{2}/2)$

The Inverse Tangent Function

Inverse Tangent Function
$\theta = \tan^{-1} x$ or $\theta = \arctan x$ means that $x = \tan\theta$ and $-\pi/2 < \theta < \pi/2$.

Why do we look in quadrants I and IV for the tangent inverse function? Note that $\tan(\pi/2)$ is undefined. For that reason, it would would be awkward to look in quadrant I and quadrant II. We would have to jump over and undefined point.

Example Evaluate each expression.

1. $\tan^1 1$
2. $\tan^{-1}(-1)$
3. $\tan^{-1} 0$
4. $\tan^{-1} -\sqrt{3}$
5. $\tan^{-1} \sqrt{3}/3$

Inverses of Cotangent, Secant, and Cosecant

When defining the inverse cotangent, secant, and cosecant, there is more than one way to restrict the range values. We will use the most common restrictions.

Inverse Cotangent, Secant, and Cosecant Functions

- $\theta = \cot^{-1} x$ or $\theta = \text{arccot}\ x$ means that $x = \cot\theta$ and θ in $(0, \pi)$.
- $\theta = \sec^{-1} x$ or $\theta = \text{arcsec}\ x$ means that $x = \sec\theta$ and θ in $[0, \pi/2) \cup (\pi/2, \pi]$.
- $\theta = \csc^{-1} x$ or $\theta = \text{arccsc}\ x$ means that $x = \csc\theta$ and θ in $[-\pi/2, 0) \cup (0, \pi/2]$.

Example Find the *degree measure* of θ in the following.

1. $\theta = \arctan 1$
2. $\theta = \sec^{-1} 2$

These are some identities that involve the inverse trig functions. We will not go into detail here.

Identities for the Functions

- $\csc^{-1}(x) = \sin^{-1}(1/x)$ for $|x| \geq 1$
- $\sec^{-1}(x) = \cos^{-1}(1/x)$ for $|x| \geq 1$
- $\cot^{-1}(x) = \begin{cases} \tan^{-1}(1/x) & \text{for} \quad x > 0 \\ \tan^{-1}(1/x) + \pi & \text{for} \quad x < 0 \\ \pi/2 & \text{for} \quad x = 0 \end{cases}$

Example

1. Find y in radians if $y = \csc^{-1}(3)$
2. Find θ in degrees if $\theta = \text{arccot}(-0.3541)$.

Example Evaluate each expression without using a calculator.

1. $\tan\left(\arcsin\frac{4}{5}\right)$
2. $\cos\left(\arctan\frac{15}{8}\right)$

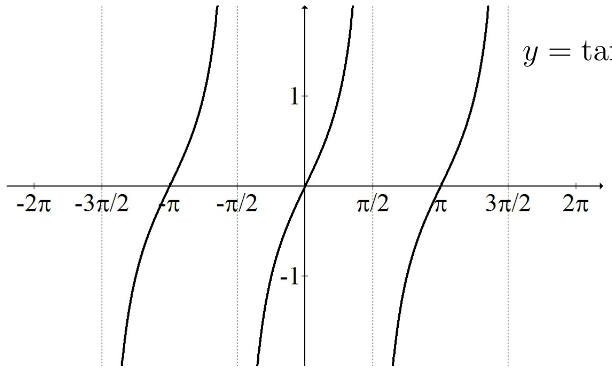
Example Write the trigonometric expression as an algebraic expression in x. Assume that x is positive.

$$\sin\left(\tan^{-1}\frac{x}{3}\right)$$

The Graphs of Inverse Trigonometric Functions

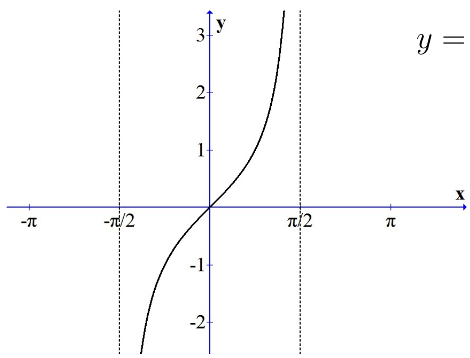
We will sketch the graph of the inverse tangent function because it is an important function with a nice graph. The graphs of the other inverse trig functions are shown in the book.

The function $y = \tan x$ is not a one-to-one function. We can see that its graph does not pass the horizontal line test.



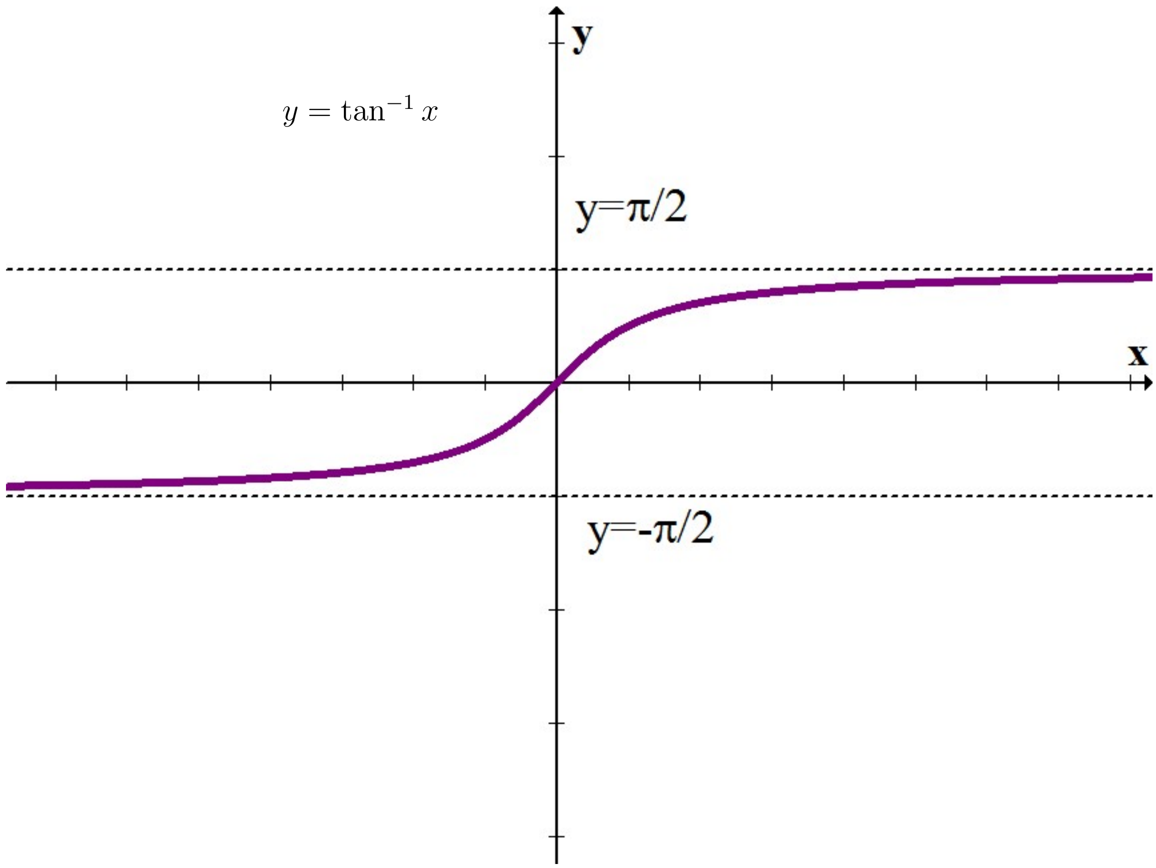
$y = \tan x$

However, if the function is restricted to the interval $-\pi/2 < x < \pi/2$, then the function passes the horizontal line test and it is one to one. Therefore, on this interval, the function $y = \tan x$ has an inverse function denoted $y = \tan^{-1} x$.



$y = \tan x,\ -\pi/2 < x < \pi/2$

The graph of an inverse function is found by reflecting the graph of the original function about the line $y = x$. The vertical asymptotes for $y = \tan x$ become horizontal asymptotes for $y = \tan^{-1} x$.


y
$y = \tan^{-1} x$
y=π/2
x
y=-π/2