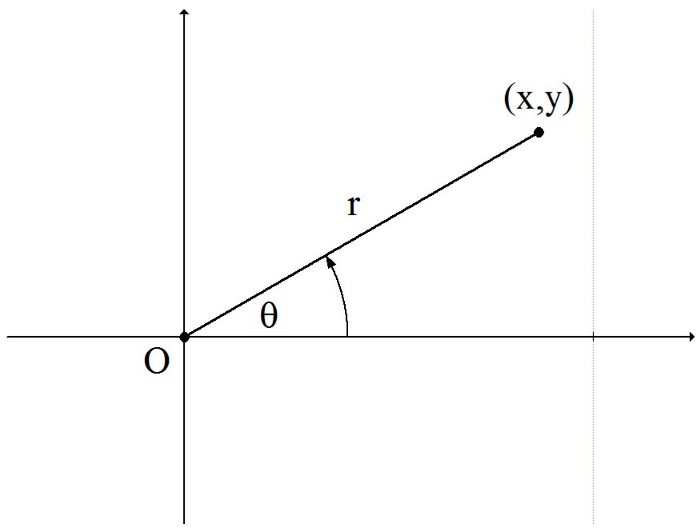
7.6 Polar Coordinates and Polar Graphs

Polar Coordinates

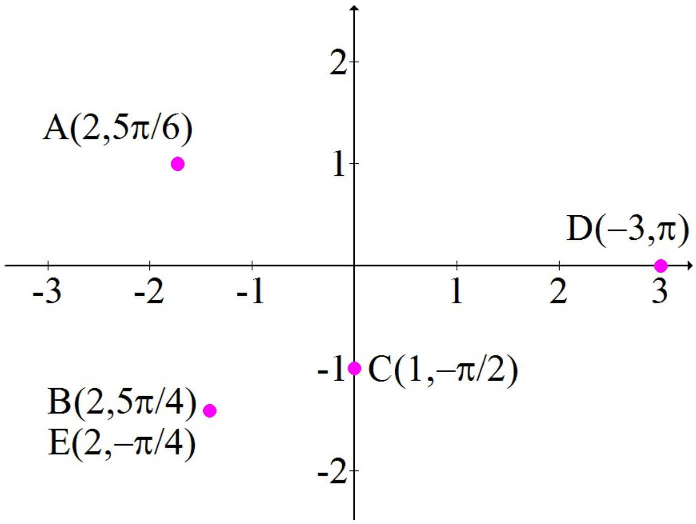
Up to this point, we have been using the **rectangular coordinate system** to describe points in a plane. A point is given by two coordinates, (x, y). The x-coordinate gives the position relative to the x-axis, and the y-coordinate gives the position relative to the the y-axis.



There is another way to describe a point P in the plane. In the **polar coordinate system**, a point P in the plane is described by two coordinates, (r, θ). The first coordinate r, is the distance of the point P from the origin. The second coordinate, θ, is the angle whose initial side is the positive x-axis and whose terminal side is the ray starting at the origin and passing through the point P. In the polar coordinate system, the origin is called the **pole**.

Example Plot the points whose polar coordinates are $A(2, 5\pi/6)$, $B(2, 5\pi/4)$, $C(1, -\pi/2)$, $D(-3, \pi)$, $E(-2, \pi/4)$.

SOLUTION



Identities

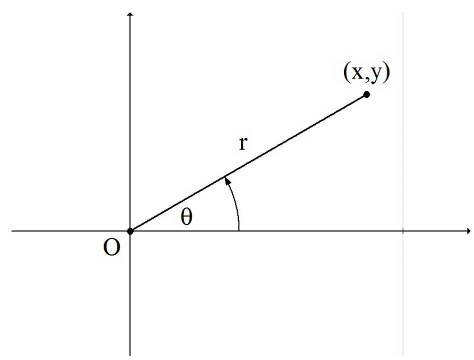
By definition, $\cos\theta = \frac{x}{r}$, and $\sin\theta = \frac{y}{r}$.
We can rewrite these as

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

Using the Pythagorean Theorem, we have

$$r^2 = x^2 + y^2$$

We also have the definition of tangent, $\tan\theta = \dfrac{y}{x}$.
Using these identities, we can convert from polar to rectangular coordinates, and vice versa.



Example Covert the following point $(4, \pi/4)$ from polar coordinates, (r, θ), to rectangular coordinates, (x, y).

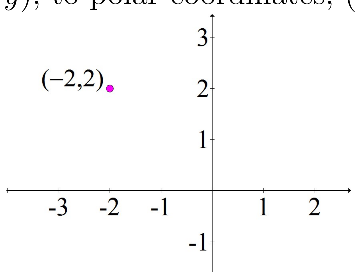
SOLUTION $r = 4,\ \theta = \pi/4$.

$$x = r\cos\theta = 4\cos\pi/4 = 4(\sqrt{2}/2) = 2\sqrt{2}$$

$$y = r\sin\theta = 4\sin\pi/4 = 4(\sqrt{2}/2) = 2\sqrt{2}$$

Answer: $(\sqrt{2}/2, \sqrt{2}/2)$.

Example Covert the following point $(-2, 2)$ from rectangular coordinates, (x, y), to polar coordinates, (r, θ). SOLUTION First plot the point.



We have $x = -2$, $y = 2$. We first find r.

$$r^2 = x^2 + y^2 = (-2)^2 + (2)^2 = 4 + 4 = 8,$$

$$r = 2\sqrt{2}$$

Now we find θ.

$$x = r\cos\theta$$

$$-2 = 2\sqrt{t}\cos\theta$$

$$-\frac{2}{2\sqrt{2}} = \cos\theta$$

$$\cos\theta = -\frac{\sqrt{2}}{2}$$

We can see that θ lies in quadrant II and that the reference angle is $\cos^{-1}\sqrt{2}/2 = \pi/4$. Therefore $\theta = 3\pi/4$.

Answer: $(2, 3\pi/4)$.

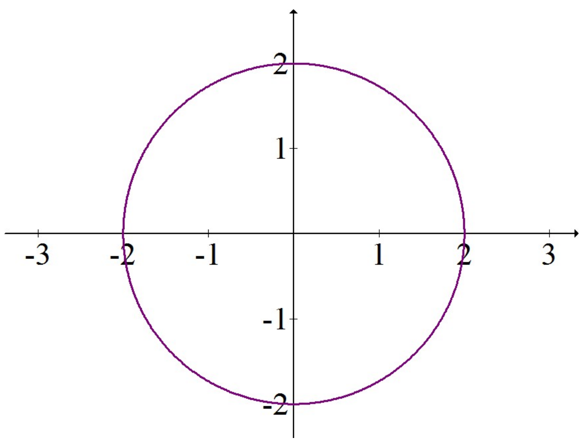
Polar Graphs

It is sometimes more convenient to express the equation of a curve in polar coordinates rather than in rectangular coordinates. Let's jump in and sketch the graphs of some polar equations.

Example Sketch the graph of the polar equation.

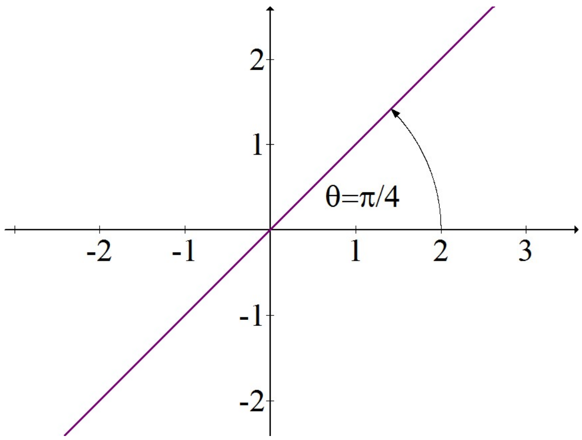
1. $r = 2$

   SOLUTION The graph is the set of all points a distance of 1 from the pole (origin), that is, a circle of radius 2 centered at the origin.

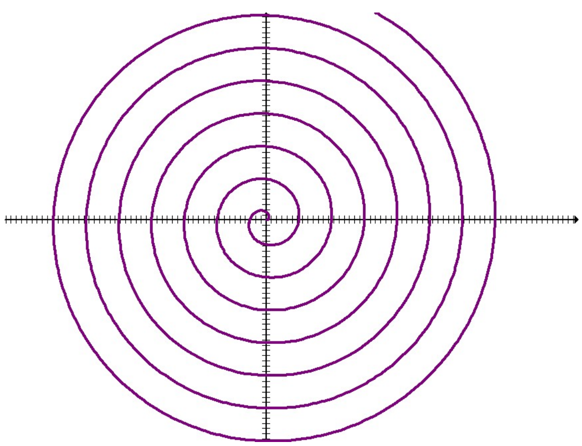
2. $\theta = \pi/4$

   SOLUTION The graph is a line that makes a $45°$ angle with the x-axis, that is, the line $y = x$.



3. $r = \theta, \quad \theta \geq 0$

   SOLUTION The graph is a spiral.



4. $r = 2\cos\theta$

   SOLUTION This equation needs to be treated in a very particular way that is different from the other equations. First multiply both sides by r.

$$r^2 = 2r\cos\theta$$

We then use the identities $r^2 = x^2 + y^2$ and $x = r\cos\theta$ to get

$$x^2 + y^2 = 2x$$

$$x^2 - 2x + y^2 = 0$$

We then complete the square, adding 1 to both sides.

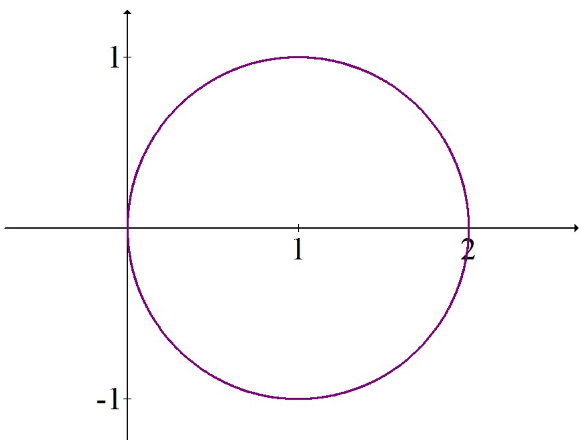
$$(x^2 - 2x + 1) + y^2 = 0 + 1$$

$$(x - 1)^2 + y^2 = 1$$

The equation of the circle in Cartesian coordinates with center (h, k) and radius r is

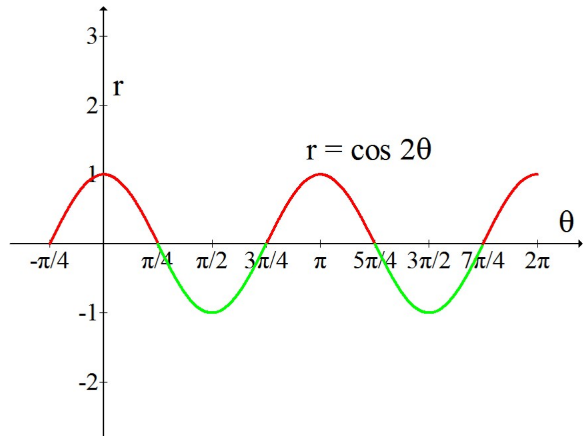
$$(x - h)^2 + (y - k)^2 = r^2$$

Therefore, the graph is a circle with center $(1, 0)$ and radius 1.

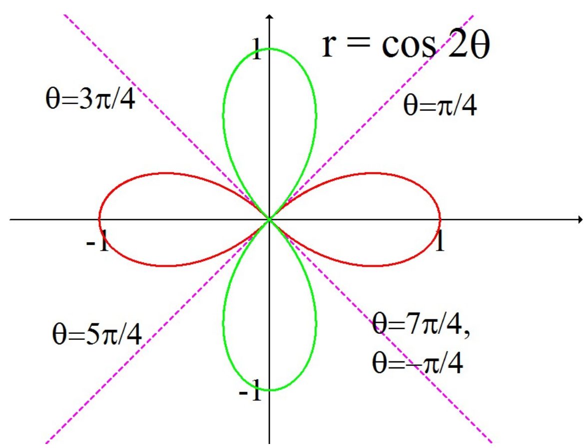


5. $r = \cos 2\theta$

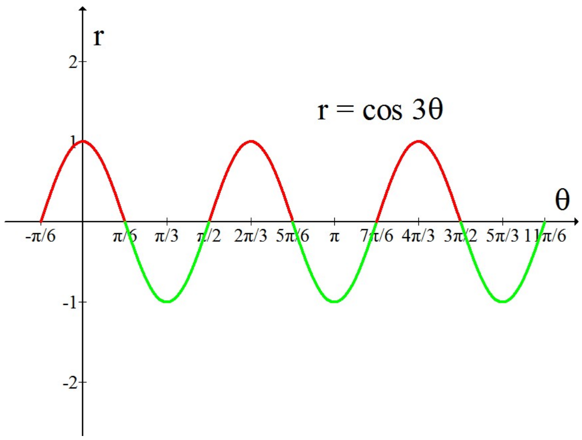
This first graph is optional. We use this graph to sketch the final answer.

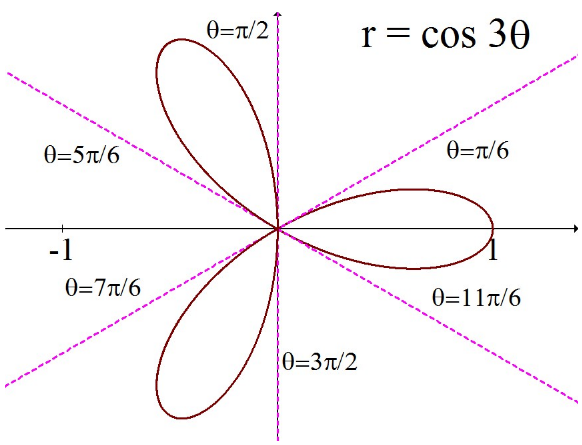


The final answer is called a four-leaf rose.

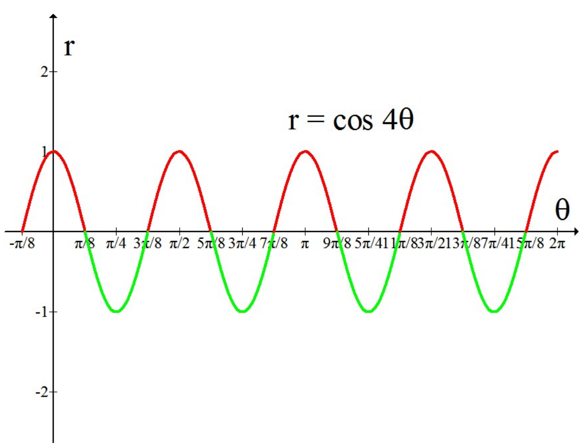


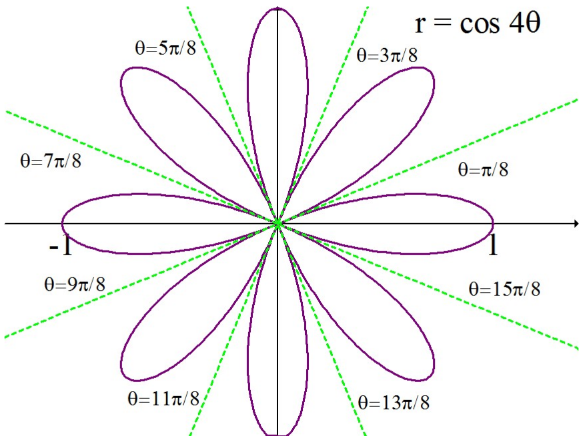
6. $r = \cos 3\theta$



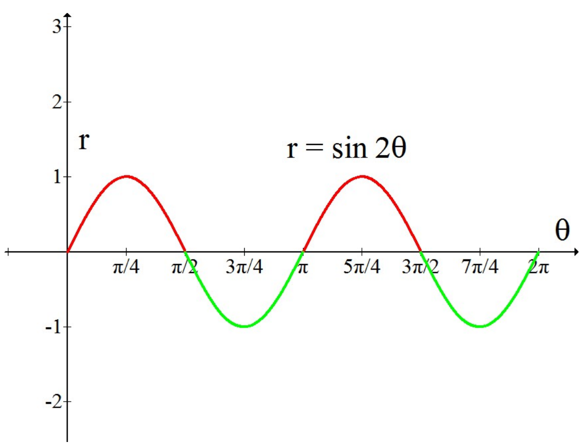


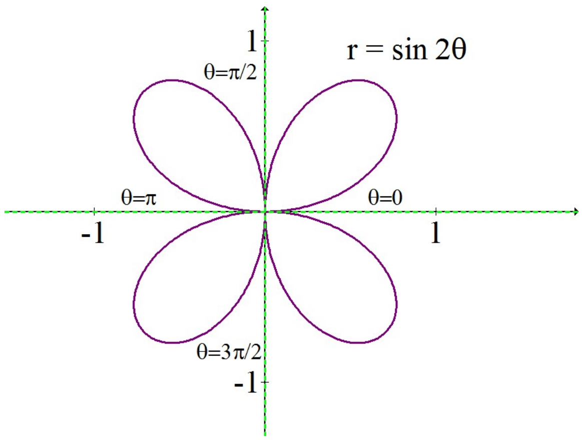
7. $r = \cos 4\theta$



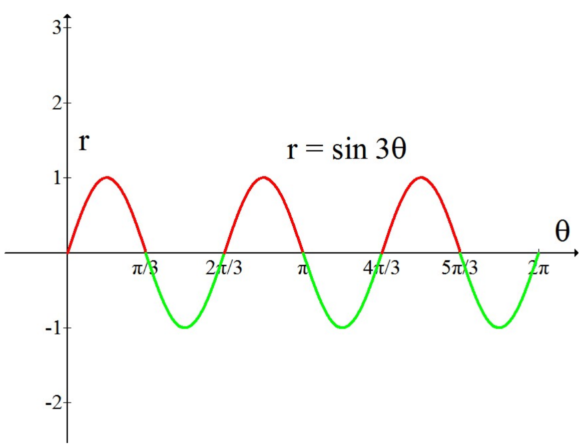


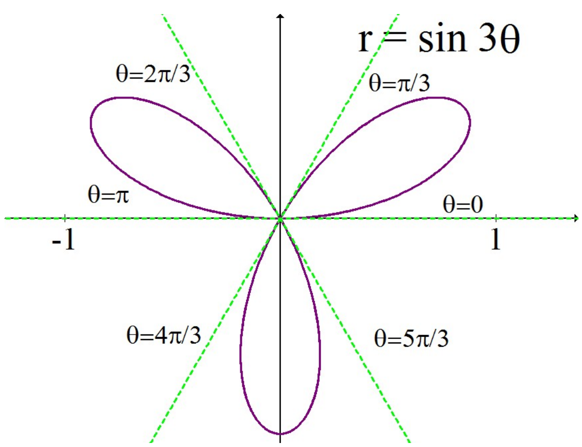
8. $r = \sin 2\theta$



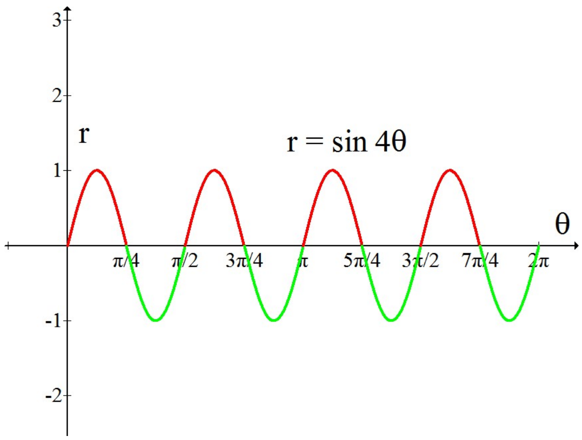


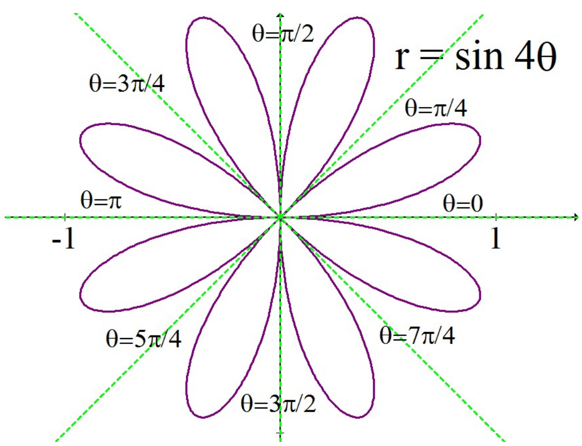
9. $r = \sin 3\theta$





10. $r = \sin 4\theta$





We can make a general statement. The graphs of $r = \cos n\theta$ and $r = \sin n\theta$ are roses with $2n$ leaves if n is even and n leaves if n is odd.