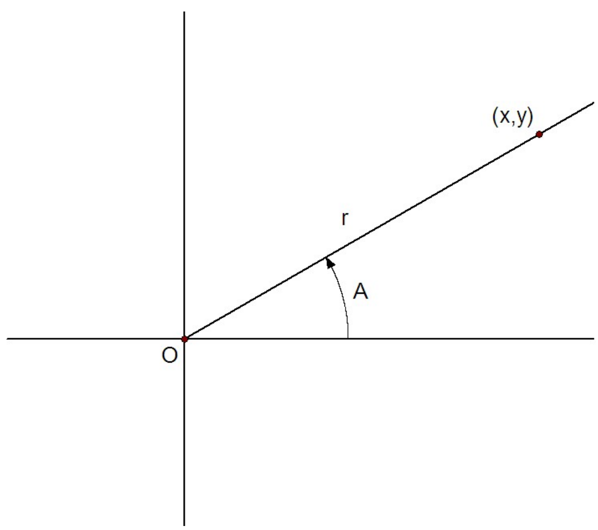
5.2 Sine and Cosine

We now place an angle A in the Cartesian plane with the vertex of the angle at the origin. One of the rays lies on the positive x-axis. Moving along an arc of the angle in a counterclockwise direction, we arrive at the other side of the angle, called the **terminal side**. Such an angle is said to be in **standard position**.

We select a point (x, y), not the origin, along the terminal side, and let

$$r = \sqrt{x^2 + y^2}.$$

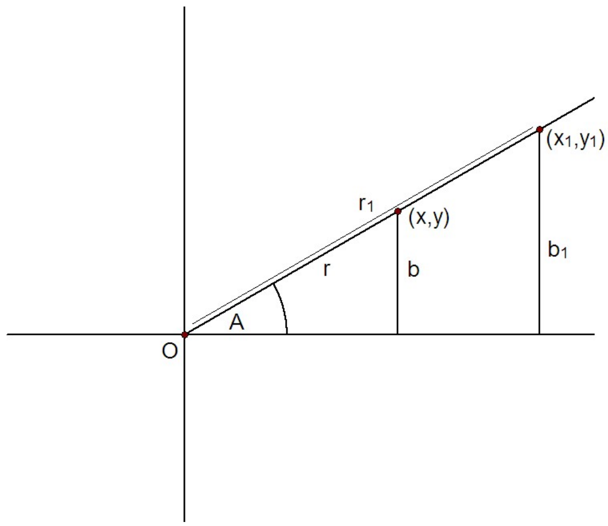


Then r is the distance from the origin $(0, 0)$ to the point (x, y). We then define

$$\text{sine } A = \frac{y}{r} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\text{cosine } A = \frac{x}{r} = \frac{x}{\sqrt{x^2 + y^2}}$$

This definition is independent of the choice of the point on the terminal side of angle A. We select another point (x_1, y_1) on the terminal side of angle A, and let $r_1 = \sqrt{x_1^2 + y_1^2}$. We see that the points (x, y) and (x_1, y_1) are corresponding vertices of two similar triangles. Similar triangles are proportional.



For our triangles, the result is that

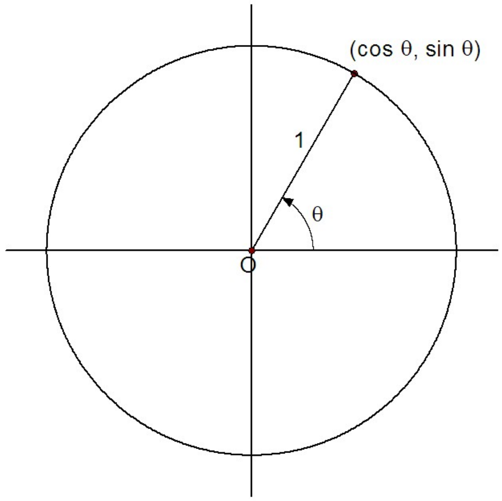
$$\frac{y}{r} = \frac{y_1}{r_1} \quad \text{and} \quad \frac{x}{r} = \frac{x_1}{r_1}.$$

This proves our definition of sine A does not depend on the choice of coordinates (x, y) on the ray.

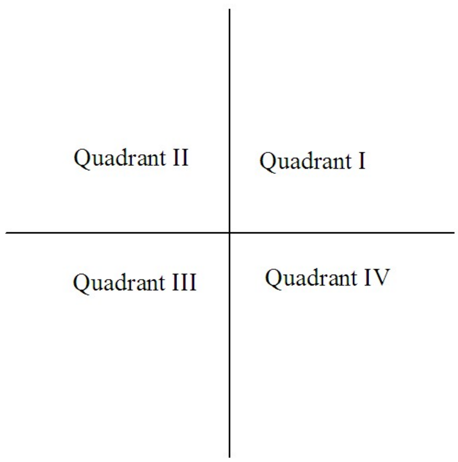
For many cases, it is convenient to select (x, y) on the circle of radius 1, so that $r = 1$. *In that case,*

$$\sin A = y \quad \text{and} \quad \cos A = x.$$

Consequently, by definition, the coordinates of a point on the circle of radius 1 are $(\cos\theta, \sin\theta)$ if θ is an angle in radians.



The Cartesian plane is partitioned into four quadrants as shown.



An angle A can be determined by a ray in any one of the four quadrants. When the ray forming the terminal side of the angle is in the first quadrant, then both sine and cosine are positive because x and y are positive. When the terminal side lies in the second quadrant, then the sine is positive because y is positive, and cosine is negative because x is negative. When the terminal side lies in the third quadrant, then sine A is negative and cosine A is also negative. When the terminal side lies in the fourth quadrant, sine A

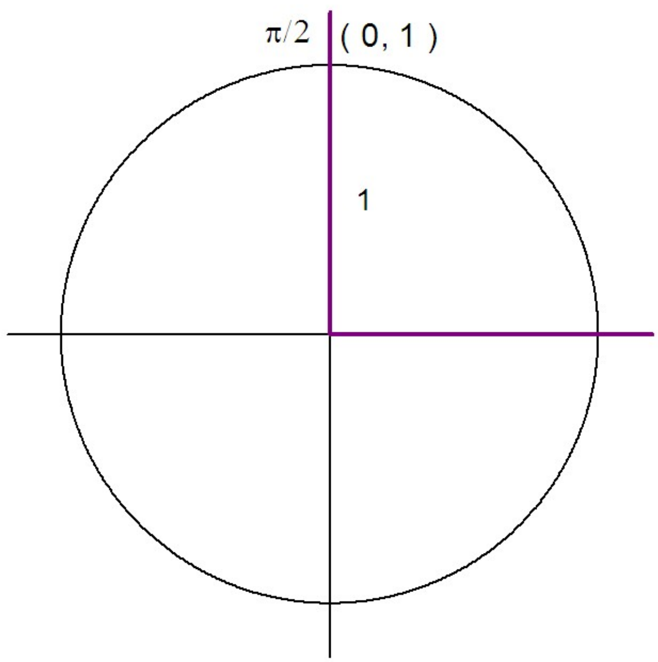
is negative and cosine A is positive.

Sine and Cosine of Angles Along the x-axis and y-axis.

Example Find $\cos \frac{\pi}{2}$.

Solution

- Locate $\pi/2$ on the unit circle, that is, the circle with radius 1.
- Write the x- and y-coordinates of the point of intersection.
- Cosine is the x-coordinate, sine is the y-coordinate.
- Answer: $\cos \frac{\pi}{2} = 0$.



Special Right Triangles

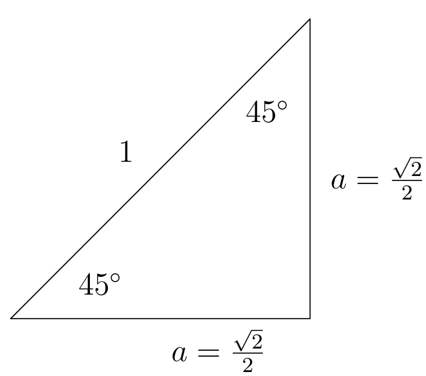
We now study two special right triangles so that we can find exact values of sine and cosine for some special angles.

The 45-45-90 **Triangle**

We set the hypotenuse equal to 1 Because this is an isosceles triangle, the legs have equal length, a. Therefore, using the Pythagorean Theorem, we have

$$a^2 + a^2 = 1$$
$$2a^2 = 1$$
$$a^2 = \frac{1}{2}$$
$$a = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Recall that 45° corresponds to $\pi/4$ radians.



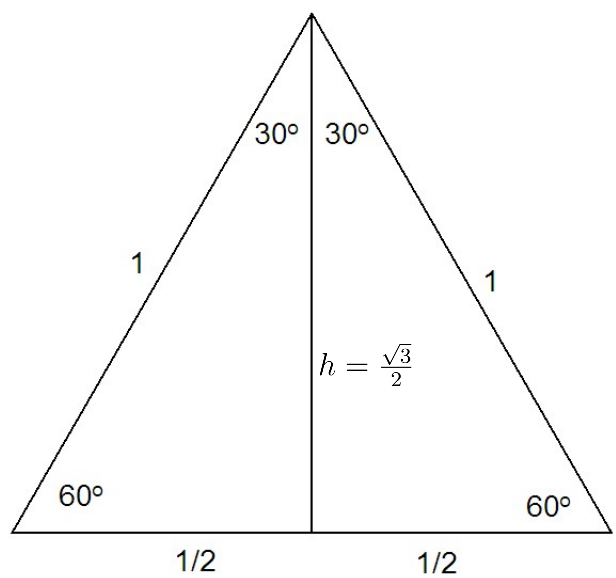
The 30-60-90 **Triangle**

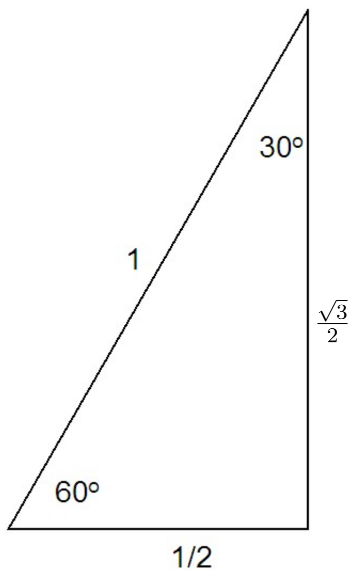
We draw an equilateral triangle with sides 1. We then bisect one of the angles. It can be shown that the bisector is perpendicular to the opposite side and that it bisects the opposite side. The result is two 30-60-90 triangles. We find the height h using the Pythagorean Theorem.

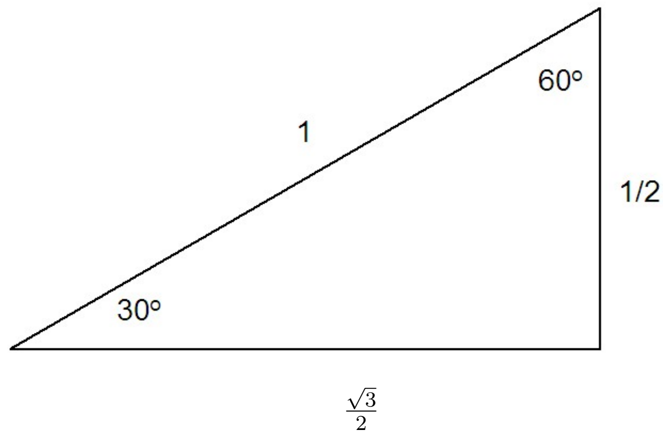
$$(1/2)^2 + h^2 = 1^2$$

$$1/4 + h^2 = 1$$
$$h^2 = 3/4$$
$$h = \sqrt{3}/2$$

Note that 60° corresponds to $\pi/3$ and 30° corresponds to $\pi/6$.







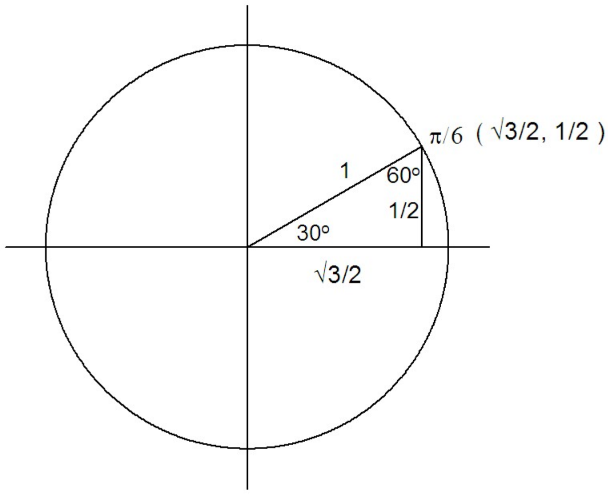
Finding Sine and Cosine of multiples of $\pi/6$, $\pi/4$, and $\pi/3$.

We can find cosine and sine of multiples of $\pi/6$, $\pi/4$, and $\pi/3$ by using our special triangles.

Example Find $\sin \frac{\pi}{6}$ and $\cos \frac{\pi}{6}$.

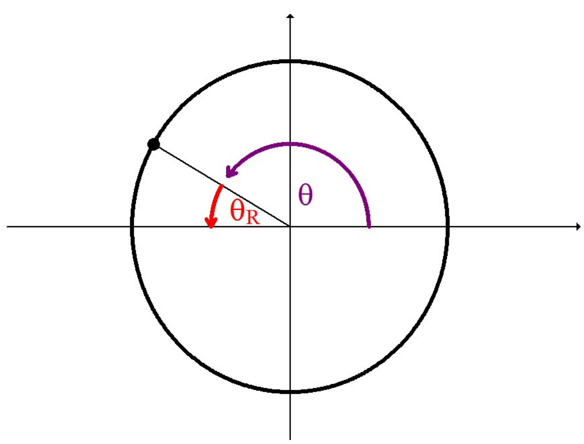
Solution

- We locate $\pi/6$ on the unit circle (the circle with radius 1).
- Insert a 30-60-90 triangle. Note that 30° corresponds to $\pi/$.
- Using the triangle as a reference, find the x- and y-coordinates of the point where the terminal side and the unit circle intersect.
- The x-coordinate is cosine, the y-coordinate is sine.
- Answer: $\sin \frac{pi}{6} = \frac{\sqrt{3}}{2}$, $\cos \frac{\pi}{6} = \frac{1}{2}$



Reference Angles

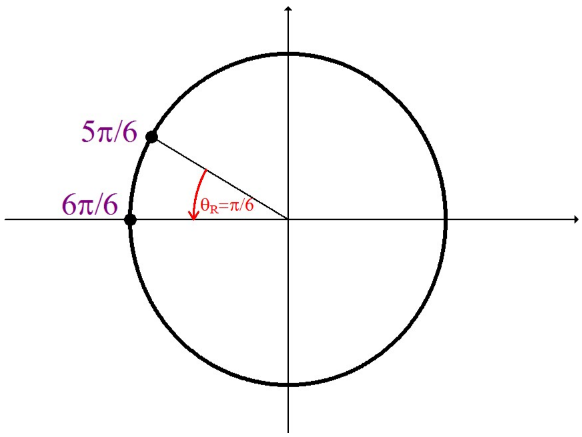
The **reference angle**, θ_R, of an angle θ is the angle which the terminal side of θ forms with the x-axis.



Example Find the reference angle θ_R corresponding to the angle $\theta = 5\pi/6$.

SOLUTION

- First locate $5\pi/6$ on the unit circle. We can use $\pi = 6\pi/6$ to help us.



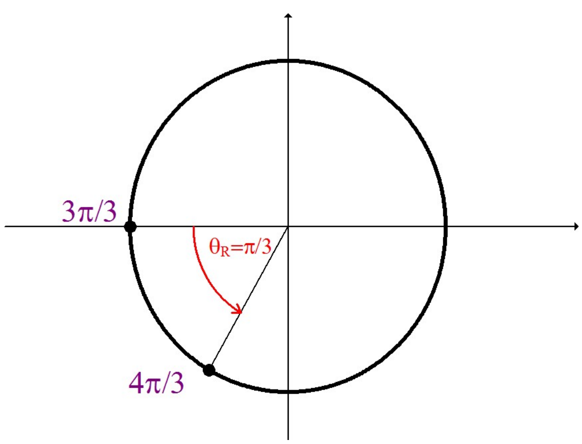
- Now we can see by looking at the graph that the reference angle is $\theta_R = \pi/6$.

We can use reference angles to find the values for sine and cosine for any multiple of $\pi/6, \pi/4, \pi/3$ or $\pi/2$.

Example Find the exact value of $\sin 4\pi/3$.

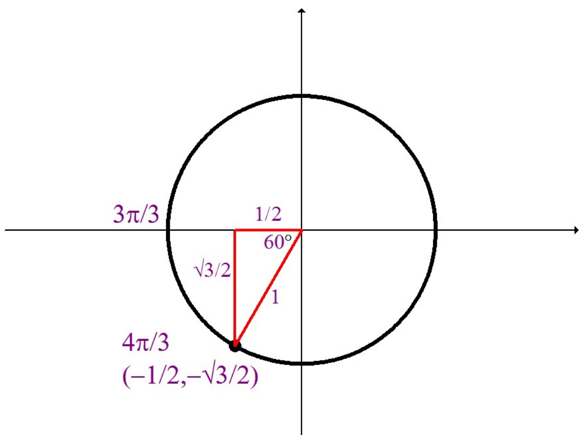
SOLUTION

- Step 1: Find $4\pi/3$ on the unit circle. Use $\pi = 3\pi/3$ as a guide.



- Step 2: Find the reference angle θ_R. We see that $\theta_R = \pi/3$, which corresponds to 60°.

- Step 3: Draw a right triangle that corresponds to a reference angle of $\theta_R = \pi/3 = 60°$. This will be a 30-60-90 triangle. Give the hypotenuse a length of 1 and label the other sides of the triangle. Keep in mind that 1/2 the smaller length and is opposite the smaller angle, 30°. The number $\sqrt{3}/2$ is the length of the longer side and is opposite the greater angle 60°.



- The value $\sin \frac{4}{3}\pi$ is the $y-$coordinate of the intercepted point. Therefore

$$\sin 4\pi/3 = -\sqrt{3}2$$

We see that $\sin 4\pi/3 = -\sqrt{3}/2$ and that $\sin \pi/3 = \sqrt{3}/2$.

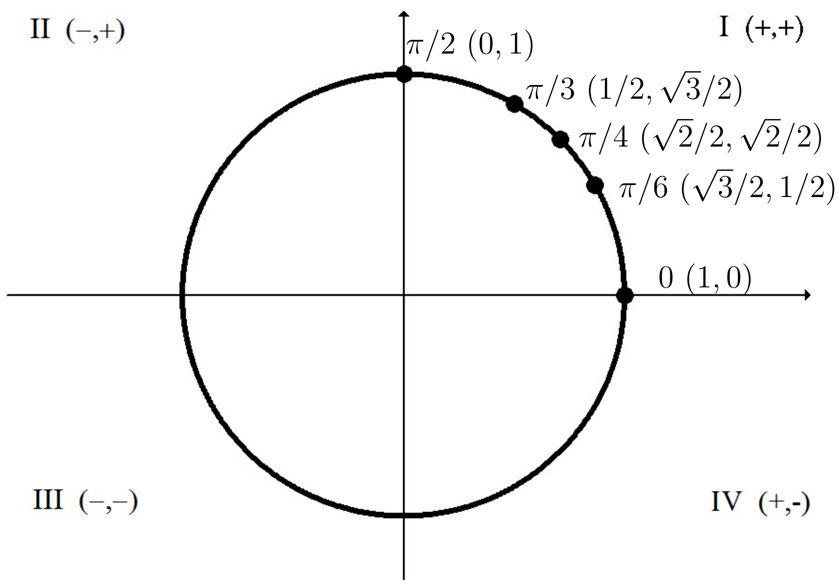
In general, either $\sin\theta = \sin\theta_R$ or $\sin\theta = -\sin\theta_R$.

By looking at the quadrant in which θ lies, one can determine whether $\sin\theta$ is positive or negative.

The same is true for $\cos\theta$. Either $\cos\theta = \cos\theta_R$ or $\cos\theta = -\cos\theta_R$.

The reference angle, θ_R, is always an angle between 0 and $\pi/2$ and therefore θ_R lies in the quadrant I when the angle is in standard position. To find the value of $\sin\theta$, some people prefer to find $\sin\theta_R$ by looking at the first quadrant and then determining whether $\sin\theta$ is positive or negative by looking at the quadrant in which θ lies. We can refer to the chart below to find the values of sine and cosine for special angles in the first quadrant.

θ deg	$0°$	$30°$	$45°$	$60°$	$90°$
θ rad	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\cos\theta$	1	$\sqrt{3}2$	$\sqrt{2}/2$	$1/2$	0
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1



As θ increases from 0 to $\pi/2$, the values of $\cos\theta$ decrease. One way to look at the sequence is as

$$\frac{\sqrt{4}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{0}}{2}$$

As θ increases from 0 to $\pi/2$, the values of $\sin\theta$ increase. The sequence of numbers is the same, but it goes in the opposite direction.

$$\frac{\sqrt{0}}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$$