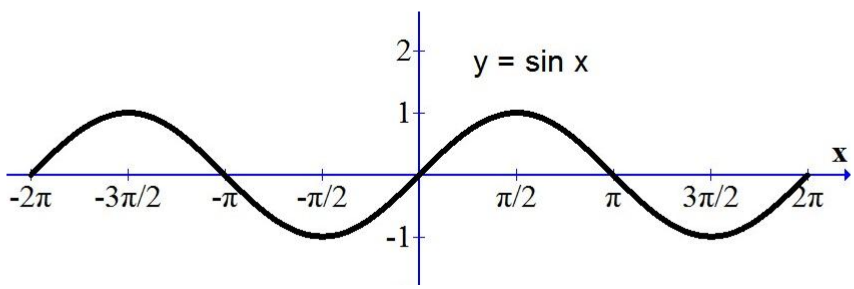
5.3 Graphs of the Sine and Cosine Functions

If $y = f(x)$ is a function and a is a nonzero constant such that $f(x) = f(x+a)$ for every x in the domain of f, then f is called a **periodic** function. The smallest such positive constant a is **period** of the function.

The Graph of the Sine Function

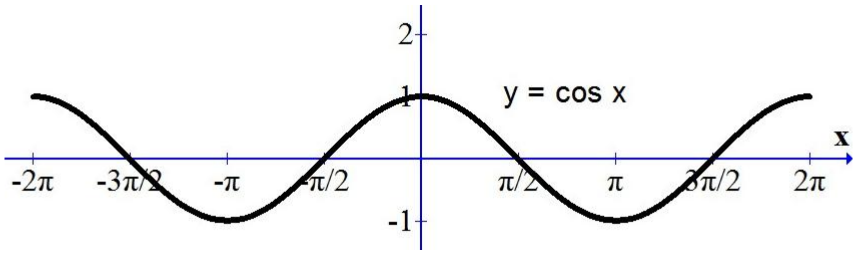
The sine function looks like a wave that passes through the origin. The graph of $y = \sin x$ or any transformation of $y = \sin x$ is called a **sine wave**, a **sinusoidal wave**, or a **sinusoid**.



- Domain: All Real Numbers.
- Range: $[-1, 1]$
- Period: $P = 2\pi$
- Quarter Period: $2\pi/4 = \pi/2$.
- The Fundamental Cycle: $[0, 2\pi]$
- Symmetry: The graph of $y = \sin x$ is symmetric about the origin. The function $y = \sin x$ is an odd function. That is, $\sin(-x) = -\sin x$.

The Graph of the Cosine Function

The cosine function looks like a wave that passes through the point $(0, 1)$.



- Domain: All Real Numbers.
- Range: $[-1, 1]$
- Period: $P = 2\pi$
- Quarter Period: $2\pi/4 = \pi/2$.
- The Fundamental Cycle: $[0, 2\pi]$
- Symmetry: The graph of $y = \cos x$ is symmetric about the y-axis. The function $y = \cos x$ is an even function. That is, $\cos(-x) = \cos x$.

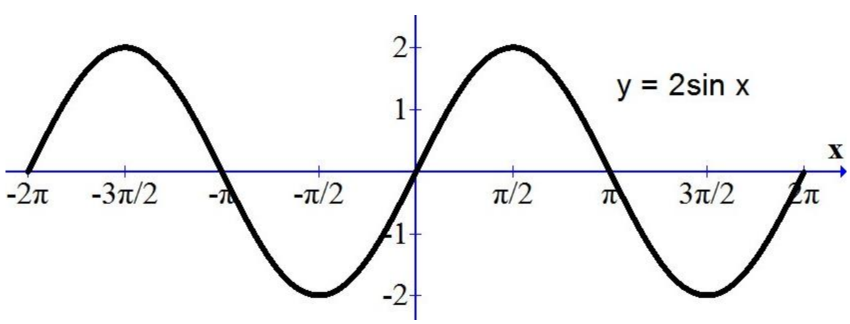
Changes in Amplitude

Multiplying a function by a positive number a will stretch the graph vertically if $a > 1$ and compress the graph vertically if $0 < a < 1$.

Example The Graph of $y = a \sin x$. Graph the following functions.

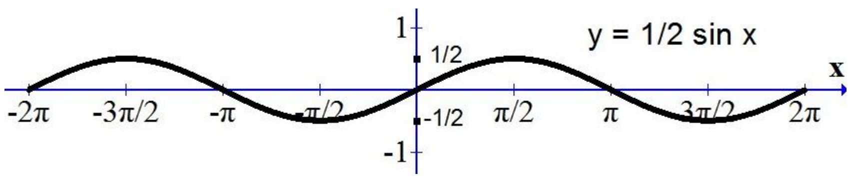
1. $y = 2 \sin x$

   Solution The amplitude is $A = 2$. The draw the sine graph, but now the maximum is 2 and the minimum is -2.



2. $y = \frac{1}{2} \sin x$

   SOLUTION The amplitude is $A = 1/2$. The draw the sine graph, but now the maximum is $1/2$ and the minimum is $-1/2$.



The **amplitude** of a sine wave, or the amplitude of the function, is the absolute value of half the distance between the maximum and minimum y-coordinates on the wave.

The amplitude of $y = a \sin x$ or $y = a \cos x$ is $|a|$.

Changes in Period

The period P of $y = \sin bx$ or $y = \cos bx$ is given by

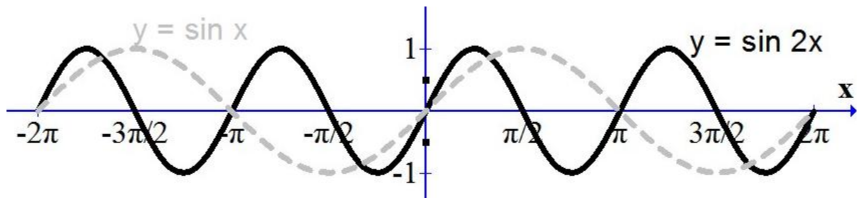
$$P = \frac{2\pi}{b}$$

I call the **quarter period** the value obtained by divided P by 4.

For any function $y = f(x)$, multiplying x by a positive number b, giving $y = f(bx)$, will stretch the graph horizontally if $0 < b < 1$, and compress the graph horizontally if $1 < b$.

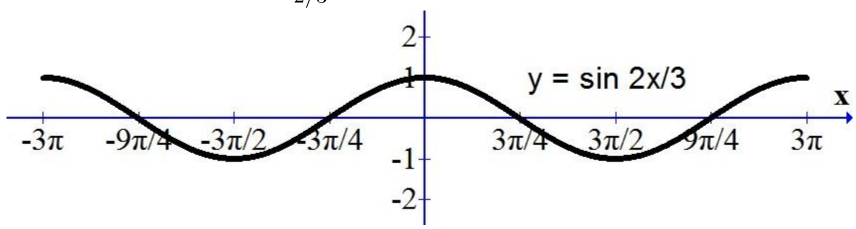
Example Graph $y = \sin 2x$, and compare to the graph of $y = \sin x$. Sketch two cycles of the graph and label the quarter interval points.

SOLUTION Period $P = 2\pi/2 = \pi$. The quarter period is $P/4 = \pi/4$.



Example Graph $y = \cos \frac{2}{3}x$. Sketch two cycles of the graph and label the quarter interval points.

SOLUTION Period $P = \frac{2\pi}{2/3} = 3\pi$. The quarter period is $P/4 = 3\pi/4$.

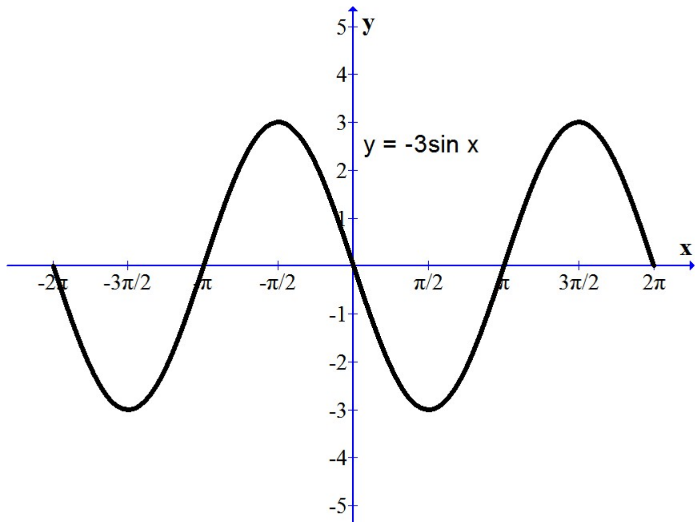


Reflection about the x-axis.

Multiplying any function, $y = f(x)$, by -1, and thus resulting in $y = -f(x)$, will reflect the graph of the function about the x-axis.

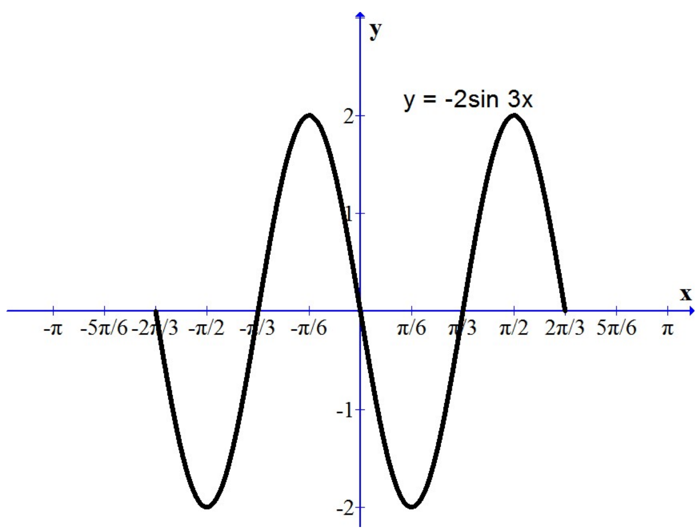
Example Sketch the graph of $y = -3 \sin x$. Sketch at least two periods and label the quarter interval points.

SOLUTION The amplitude is $A = |-3| = 3$. The graph is reflected about the x-axis. The period is $P = 2\pi$. The quarter period is $P/4 = \frac{\pi}{2}$.



Example Graph $y = -2\sin 3x$. Sketch at least two cycles of the function and label the quarter interval points.

Solution The amplitude is $A = |-2| = 2$. The graph is reflected about the x-axis. The period is $P = 2\pi/3$. The quarter period is $P/4 = \frac{1}{4} \cdot \frac{2\pi}{3} = \frac{\pi}{6}$.



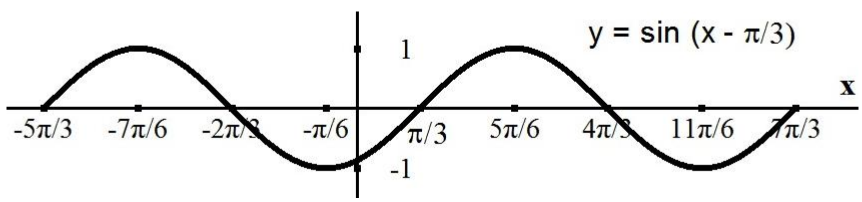
Phase Shifts

For any function $y = f(x)$, if x is replaced by $x - d$, resulting in the function $y = f(x - d)$, then the graph of f is shifted to the right if $d > 0$ and to the left if $d < 0$. For trigonometric functions, a translation to the right or the left is called a **phase shift**.

Example Sketch two cycles of the graph of $y = \sin(x - \pi/3)$. Label the quarter period points.

SOLUTION

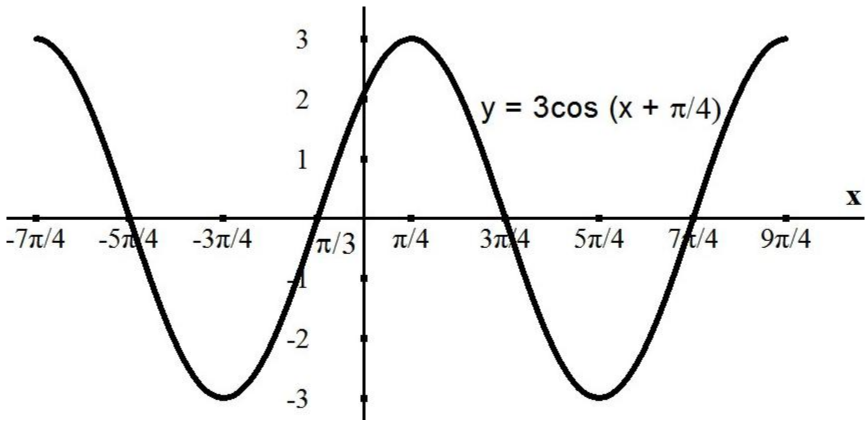
Amplitude:	$A = 1$
Period:	$P = 2\pi$
Qtr Period	$P/4 = \pi/2$
Phase shift:	$\varphi = \pi/3$, shift right



Example Sketch two cycles of the graph of $y = 3\cos(x + \pi/4)$. Label the quarter period points.

SOLUTION

Amplitude:	$A = 3$
Period:	$P = 2\pi$
Qtr Period	$P/4 = \pi/2$
Phase shift:	$\varphi = -\pi/4$, shift left

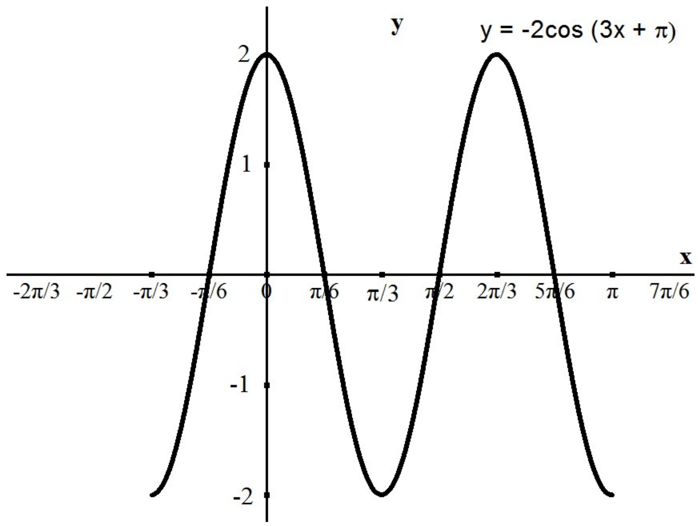


Example Sketch two cycles of the graph of $y = -2\cos(3x + \pi)$. Label the quarter period points.

SOLUTION

$$y = -2\cos 3\left(x + \frac{\pi}{3}\right)$$

Amplitude:	$A = 2$
Reflection:	About the x-axis
Period:	$P = 2\pi/3$
Qtr Period	$P/4 = \pi/6$
Phase shift:	$\varphi = \pi/3$, shift left



Vertical Translations

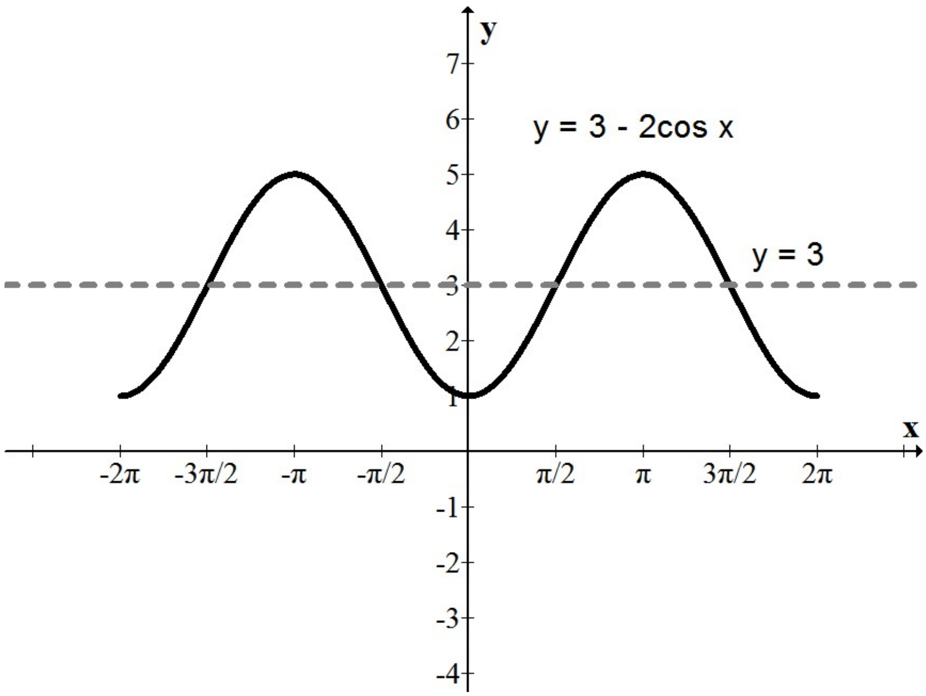
For the graphs of sine waves, I call the **axis of the function** the horizontal line, $y = L$, where L is the average value of the maximum and the minimum of the function. For $y = \sin x$ and $y = \cos x$, the axis of the function is just the x-axis. I don't think that anybody else uses this terminology.

Adding a number c to any function $y = f(x)$, resulting in the function $y = c + f(x)$, will shift the graph of f upward if $c > 0$ and downward if $c < 0$.

Example Sketch two cycles of the graph of $y = 3 - 2\cos x$. Label the quarter period points.
SOLUTION

$$y = -2\cos(x) + 3$$

Amplitude:	$A = 2$
Reflection:	About the x-axis
Period:	$P = 2\pi$
Qtr Period	$P/4 = \pi/2$
Phase shift:	none
Vertical Shift:	Up by 3



Guidelines for Sketching the Graphs of Sine and Cosine Functions

To sketch the graph of $y = c + a \sin b(x - d)$ or $y = a \cos b(x - d)$, with $b > 0$, follow these steps.

Step 1: Find the period. Use $P = 2\pi/b$, where b is the coefficient of x.

Step 2: Find the quarter period by dividing the period by 4. QtrPer= $P/4$.

Step 3: Find the phase shift, d. This is the starting point of the graph.

Step 3: Sketch two cycles of the the wave curve. The sine graph starts at the x-axis, while the cosine graph starts at its maximum.

Step 4: Label the starting point. This is the phase shift d.

Step 5 Add the quarter period to the starting point. If necessary, get a common denominator for the starting point and the quarter period in order to make the addition easier. Repeat to label the quarter interval points.

Step 4: Label the maximum and minimum values of the curve using the amplitude $|a|$.

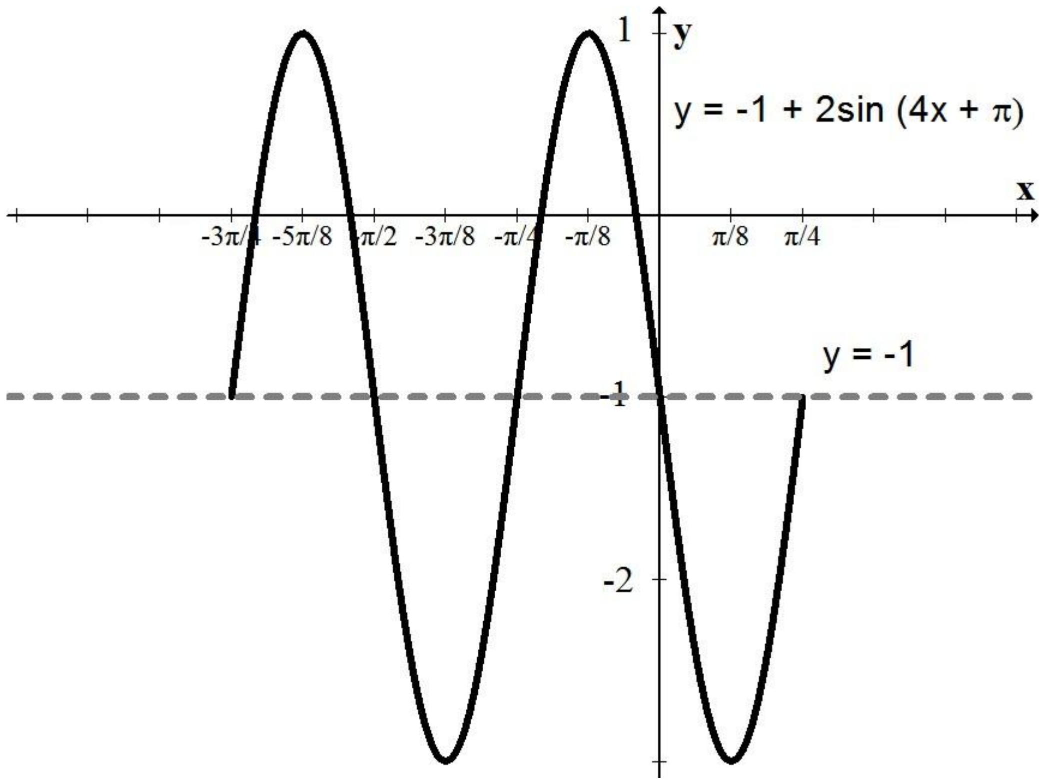
Step 5: If the graph has been shifted up or down by c, then draw a dotted line for $y = c$. Sketch the graph of the function as if the line $y = c$ is the x-axis. Then later go back and draw the x-axis.

Combinations of Translations

Example Sketch two cycles of the graph of $y = -1 + 2\sin(4x + \pi)$. Label the quarter period points.
SOLUTION

$$y = 2\sin 4\left(x + \frac{\pi}{4}\right) + -1$$

Amplitude:	$A = 2$
Reflection:	none
Period:	$P = 2\pi/4 = \pi/2$
Qtr Period	$P/4 = \pi/8$
Phase shift:	$\varphi = -\pi/4$, shift left
Vertical Shift:	down by 1


y
1
y = -1 + 2sin (4x + π)
x
-3π/4
-5π/8
-π/2
-3π/8
-π/4
-π/8
π/8
π/4
y = -1
-1
-2