6.6 Conditional Trig Equations

1. Find all real solutions in the interval $[0, 2\pi)$. Give exact values, not decimal approximations.

   (a) $\cos x = 0$ ANSWER: $\{\pi/2, 3\pi/2\}$.

   (b) $\sin x = 1$ ANSWER: $\{\pi/2\}$

   (c) $\cos x = \dfrac{\sqrt{2}}{2}$
   ANSWER: $\{\pi/4, 7\pi/4\}$

2. Find all solutions in degrees in the interval $[0, 360°)$.

   (a) $\sin x = \dfrac{\sqrt{3}}{2}$ ANSWER: $\{\pi/3, 2\pi/3\}$

   (b) $\cos x = -\dfrac{\sqrt{3}}{2}$ ANSWER: $\{5\pi/6, 7\pi/6\}$

3. Find all real solutions in the interval $[0, 2\pi)$. Use a calculator and round your answer to two decimal places.

   (a) $\sin\alpha = -0.321$

   SOLUTION $\sin\alpha$ is negative for α in quadrants III and IV. The reference angle is $\alpha_R = \sin^{-1} 0.321 \approx 0.33$. The answer is

   $$\alpha_1 = \pi + \alpha_R = \pi + \sin^{-1} 0.321 \approx \pi + 0.33 \approx 3.47$$

   $$\alpha_2 = 2\pi - \alpha_R = 2\pi - \sin^{-1} 0.321 \approx 2\pi - 0.33 \approx 5.96$$

   (b) $\cos\alpha = -0.75$

   SOLUTION $\cos\alpha$ is negative for α in quadrants II and III. The reference angle is $\alpha_R = \cos^{-1} 0.75 \approx 0.72$. The answer is

   $$\alpha_1 = \pi - \alpha_R = \pi - \cos^{-1} 0.75 \approx \pi - 0.72 \approx 2.42$$

   $$\alpha_2 = \pi + \alpha_R = \pi + \cos^{-1} 0.75 \approx \pi + 0.72 \approx 3.86$$

(c) $\tan\alpha = 3.91$

SOLUTION $\tan\alpha$ is positive for α in quadrants I and III. The reference angle is $\alpha_R = \tan^{-1} 3.91 \approx 1.32$. The answer is

$$\alpha_1 = \alpha_R = \tan^{-1} 0.391 \approx 1.32$$

$$\alpha_2 = \pi + \alpha_R = \pi + \tan^{-1} 3.91 \approx \pi + 1.32 \approx 4.46$$

4. Find all real numbers that satisfy the equation. Give exact values.

   (a) $\sin\theta = -\sqrt{3}/2$ ANSWER: $\{4\pi/3 + 2k\pi,\ 5\pi/3 + 2k\pi\}$, k is an integer.

   (b) $\sec\theta = -2$ ANSWER: $\{2\pi/3 + 2k\pi,\ 4\pi/3 + 2k\pi\}$, k is an integer.

   (c) $\csc\theta = \sqrt{2}$
   ANSWER: $\{\pi/4 + 2k\pi,\ 3\pi/4 + 2k\pi\}$, k is an integer.

5. Find all angles in degrees that satisfy each equation. Give exact values.

   (a) $\cos\alpha = -1$ ANSWER: $\{180° + 360°k\}$, k is an integer.

   (b) $\tan\alpha = 1$ ANSWER: $\{45° + 180°k\}$, k is an integer.

   (c) $\sin\alpha = -\sqrt{2}/2$ ANSWER: $\{225° + 360°k,\ 315° + 360°k\}$, k is an integer.

6. Find all real solutions in the interval $[0, 2\pi)$. Give exact values.

   (a) $2\cos^2\theta = 1 - \cos\theta$

   SOLUTION

$$2\cos^2\theta = 1 - \cos\theta$$

$$2\cos^2\theta + \cos\theta - 1 = 0$$

Let $u = \cos\theta$.

$$2u^2 + u - 1 = 0$$

$$(2u - 1)(u + 1) = 0$$

$$u = 1/2, \ u = 1$$
$$\cos\theta = 1/2, \ \cos\theta = -1$$

ANSWER: $\{\pi/3, \pi, \ 5\pi/3\}$.

(b) $4\cos^2\theta - 1 = 0$

SOLUTION

$$4\cos^2\theta = 1$$
$$\cos^2\theta = 1/4$$
$$\cos\theta = \pm 1/2$$

ANSWER: $\{\pi/6, \ 5\pi/6, \ 7\pi/6, \ 11\pi/6\}$.

(c) $2\cos^2 x + 5\cos x + 2 = 0$

SOLUTION

$$2\cos^2 x + 5\cos x + 2 = 0$$

Let $u = \cos x$

$$2u^2 + 5u + 2 = 0$$
$$(2u+1)(u+2) = 0$$
$$u = -1/2, \ u = -2$$
$$\cos x = -1/2, \ \cos x = -2$$

Note that $\cos x = -2$ has no solutions because the minimum value that cosine can take is -1.

ANSWER: $\{2\pi/3, \ 4\pi/3\}$.

7. Find all real numbers that satisfy the equation.

(a) $\cos\left(\frac{x}{2}\right) = \frac{1}{2}$

SOLUTION

$$\begin{array}{ll} \frac{x}{2} = \frac{\pi}{3} + 2k\pi, & \frac{x}{2} = \frac{5\pi}{3} + 2k\pi \\ x = 2\left(\frac{\pi}{3} + 2k\pi\right), & x = 2\left(\frac{5\pi}{3} + 2k\pi\right) \\ x = \frac{2\pi}{3} + 4k\pi, & x = \frac{10\pi}{3} + 4k\pi \end{array}$$

ANSWER: : $\{\frac{2\pi}{3} + 4k\pi, \ \frac{10\pi}{3} + 4k\pi, \ k \in \mathbb{Z}\}$

(b) $2\cos 2x = -\sqrt{2}$

SOLUTION $\cos 2x = -\frac{\sqrt{2}}{2}$

$$\begin{array}{ll} 2x = \frac{3\pi}{4} + 2k\pi, & 2x = \frac{5\pi}{4} + 2k\pi \\ x = \frac{1}{2}\left(\frac{3\pi}{4} + 2k\pi\right), & x = \frac{1}{2}\left(\frac{5\pi}{4} + 2k\pi\right) \\ x = \frac{3\pi}{8} + k\pi, & x = \frac{5\pi}{8} + k\pi \end{array}$$

ANSWER: : $\{\frac{3\pi}{8} + k\pi,\ \frac{5\pi}{8} + k\pi,\ k \in \mathbb{Z}\}$

(c) $\sin\left(\dfrac{x}{3}\right) + 1 = 0$

SOLUTION $\sin(x/3) = -1.$

$$\frac{x}{3} = \frac{3\pi}{2} + 2k\pi$$
$$x = 3\left(\frac{3\pi}{2} + 2k\pi\right)$$
$$x = \frac{9\pi}{2} + 6k\pi$$

ANSWER: $\left\{\dfrac{9\pi}{2} + 6k\pi,\ k \in \mathbb{Z}\right\}$

8. Find all real numbers in the interval $[0, 2\pi)$ that satisfy the equation.

(a) $\sin 2x = \sqrt{2}/2$

SOLUTION

$$\begin{array}{lll} & 2x = \frac{\pi}{4} + 2k\pi, & 2x = \frac{3\pi}{4} + 2k\pi \\ & x = \frac{1}{2}\left(\frac{\pi}{4} + 2k\pi\right), & x = \frac{1}{2}\left(\frac{3\pi}{4} + 2k\pi\right) \\ & x = \frac{\pi}{8} + k\pi, & x = \frac{3\pi}{8} + k\pi \\ k = 0, & x = \frac{\pi}{8} + 0\cdot\pi = \frac{\pi}{8}, & x = \frac{3\pi}{8} + 0\cdot\pi = \frac{3\pi}{8} \\ k = 1, & x = \frac{\pi}{8} + 1\cdot\pi = \frac{9\pi}{8}, & x = \frac{3\pi}{8} + 1\cdot\pi = \frac{11\pi}{8} \end{array}$$

For $k = 2, 3, 4, \ldots$ and for negative integers, the value for x is not in the interval $[0, 2\pi)$.

ANSWER: : $\left\{\dfrac{\pi}{8}, \dfrac{3\pi}{8}, \dfrac{9\pi}{8}, \dfrac{11\pi}{8}\right\}$

(b) $\tan 2x = 1$

SOLUTION

$$\begin{array}{lll} & 2x = \frac{\pi}{4} + 2k\pi, & 2x = \frac{5\pi}{4} + 2k\pi \\ & x = \frac{1}{2}\left(\frac{\pi}{4} + 2k\pi\right), & x = \frac{1}{2}\left(\frac{5\pi}{4} + 2k\pi\right) \\ & x = \frac{\pi}{8} + k\pi, & x = \frac{5\pi}{8} + k\pi \\ k = 0, & x = \frac{\pi}{8} + 0 \cdot \pi = \frac{\pi}{8}, & x = \frac{5\pi}{8} + 0 \cdot \pi = \frac{5\pi}{8} \\ k = 1, & x = \frac{\pi}{8} + 1 \cdot \pi = \frac{9\pi}{8}, & x = \frac{5\pi}{8} + 1 \cdot \pi = \frac{13\pi}{8} \end{array}$$

For $k = 2, 3, 4, \ldots$ and for negative integers, the value for x is not in the interval $[0, 2\pi)$.

ANSWER: : $\left\{\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}\right\}$

9. Find all real numbers in the interval $[0, 2\pi)$ that satisfy the equation. One of these problems will be on the test.

(a) $\sin 2x = \cos x$

SOLUTION Use the identity $\sin 2x = 2\sin x \cos x$.

$$2\sin x \cos x = \cos x$$

$$2\sin x \cos x - \cos x = 0$$

$$\cos x\,(2\sin x - 1) = 0$$

$$\cos x = 0, \quad 2\sin x - 1 = 0$$

$$\cos x = 0, \quad \sin x = 1/2$$

ANSWER: $\{\pi/6, \pi/2, 5\pi/6, 3\pi/2\}$.

(b) $\sin 2x = \sin x$

SOLUTION Use the identity $\sin 2x = 2\sin x \cos x$.

$$2\sin x \cos x = \sin x$$

$$2\sin x \cos x - \sin x = 0$$

$$\sin x\,(2\cos x - 1) = 0$$
$$\sin x = 0, \quad 2\cos x - 1 = 0$$
$$\sin x = 0, \quad \cos x = 1/2$$

ANSWER: $\{0, \pi/3, \pi, 5\pi/3\}$.

(c) $\cos 2x = \cos x$ Use the identity $\cos 2x = \cos^2 x - \sin^2 x$.

$$\cos^2 x - \sin^2 x = \cos x$$
$$\cos^2 x - \sin^2 x - \cos x = 0$$

Use $\sin^2 x = 1 - \cos^2 x$.

$$\cos^2 x - (1 - \cos^2 x) - \cos x = 0$$
$$2\cos^2 x - \cos x - 1 = 0$$

Let $u = \cos x$.

$$2u^2 - u - 1 = 0$$
$$(2u + 1)(u - 1) = 0$$
$$2u + 1 = 0, \quad u - 1 = 0$$
$$u = -1/2, \quad u = 1$$
$$\cos x = -1/2, \quad \cos x = 1$$

ANSWER: $\{0, 2\pi/3, 4\pi/3\}$.

(d) $\cos 2x = \sin x$ Use the identity $\cos 2x = \cos^2 x - \sin^2 x$.

$$\cos^2 x - \sin^2 x = \sin x$$
$$\cos^2 x - \sin^2 x - \sin x = 0$$

Use $\cos^2 x = 1 - \sin^2 x$.

$$(1 - \sin^2 x) - \sin^2 x - \sin x = 0$$
$$-2\sin^2 x - \sin x + 1 = 0$$

Let $u = \sin x$.

$$-2u^2 - u + 1 = 0$$
$$2u^2 + u - 1 = 0$$

$$(2u-1)(u+1)=0$$
$$2u-1=0, \quad u+1=0$$
$$u=1/2, \quad u=-1$$
$$\sin x=1/2, \quad \sin x=-1$$

ANSWER: $\{\pi/6, 5\pi/6, 3\pi/2\}$.