6.3 Sum and Differences Identities

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$
$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

1. Find the exact value of each expression.

   (a) $\cos\left(\frac{5}{12}\pi\right)$

$$\begin{aligned}
&= \cos\left(\frac{2\pi}{12} + \frac{3\pi}{12}\right) \\
&= \cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right) \\
&= \cos\frac{\pi}{6}\cos\frac{\pi}{4} - \sin\frac{\pi}{6}\sin\frac{\pi}{4} \\
&= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} \\
&= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\
&= \frac{\sqrt{6}-\sqrt{2}}{4}
\end{aligned}$$

   (b) $\sin\left(\frac{\pi}{12}\right)$

$$\begin{aligned}
&= \sin\left(\frac{3\pi}{12} - \frac{2\pi}{12}\right) \\
&= \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right) \\
&= \sin\frac{\pi}{4}\cos\frac{\pi}{6} - \cos\frac{\pi}{4}\sin\frac{\pi}{6} \\
&= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} \\
&= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\
&= \frac{\sqrt{6}-\sqrt{2}}{4}
\end{aligned}$$

(c) $\sin\left(\frac{7\pi}{12}\right)$

$$\begin{aligned} &= \sin\left(\frac{3\pi}{12}+\frac{4\pi}{12}\right) \\ &= \sin\left(\frac{\pi}{4}+\frac{\pi}{3}\right) \\ &= \sin\frac{\pi}{4}\cos\frac{\pi}{3}+\cos\frac{\pi}{4}\sin\frac{\pi}{3} \\ &= \frac{\sqrt{2}}{2}\cdot\frac{1}{2}+\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} \\ &= \frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2} \\ &= \frac{\sqrt{2}+\sqrt{6}}{2} \end{aligned}$$

(d) $\sin 15^\circ$

$$\begin{aligned} &= \sin(45^\circ-30^\circ) \\ &= \sin 45^\circ\cos 30^\circ-\cos 45^\circ\sin 30^\circ \\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{1}{2} \\ &= \frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4} \\ &= \frac{\sqrt{4}-\sqrt{2}}{4} \end{aligned}$$

(e)

$$\begin{aligned} \cos 105^\circ &= \cos(60^\circ+45^\circ) \\ &= \cos 60^\circ\cos 45^\circ-\sin 60^\circ\sin 45^\circ \\ &= \frac{1}{2}\cdot\frac{\sqrt{2}}{2}-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{2}}{4}-\frac{\sqrt{6}}{4} \\ &= \frac{\sqrt{4}-\sqrt{6}}{4} \end{aligned}$$

2. Write each expression as a single function of θ.

   (a) $\cos\left(\frac{\pi}{6}+\theta\right)$

$$\begin{aligned} &= \cos\frac{\pi}{6}\cos\theta - \sin\frac{\pi}{6}\sin\theta \\ &= \frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta \end{aligned}$$

   (b) $\cos\left(\frac{\pi}{4}-\theta\right)$

$$\begin{aligned} &= \cos\frac{\pi}{4}\cos\theta + \sin\frac{\pi}{4}\sin\theta \\ &= \frac{\sqrt{2}}{2}\cos\theta + \frac{\sqrt{2}}{2}\sin\theta \end{aligned}$$

   (c) $\sin\left(\frac{\pi}{3}+\theta\right)$

$$\begin{aligned} &= \sin\frac{\pi}{3}\cos\theta + \cos\frac{\pi}{3}\sin\theta \\ &= \frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta \end{aligned}$$

3. Use the cofunction identities to find an angle θ that satisfies each of the following.

   Note: Two angles are said to be **complementary** if their sum is 90°. We say that the **complement** of an angle θ is the angle $90^\circ - \theta$.

   The **cofunction identities** say

$$\text{Trig Function}(\theta) = \text{Cofunction}\,(\text{complement of }\theta)$$

   For example, $\sin\theta = \cos\,(\text{the complement of }\theta)$, i.e. $\sin\theta = \cos\,(90^\circ - \theta)$.

   (a) $\cot\theta = \tan\frac{\pi}{6}$

$$\cot\theta = \tan\frac{\pi}{6}$$

$$\begin{aligned}\cot\theta &= \tan\left(\frac{\pi}{2}-\theta\right) \quad \leftarrow \text{cofunction identity}\\ \tan\frac{\pi}{6} &= \tan\left(\frac{\pi}{2}-\theta\right)\\ \frac{\pi}{6} &= \frac{\pi}{2}-\theta\\ \theta &= \frac{\pi}{2}-\frac{\pi}{6}=\frac{\pi}{3}\end{aligned}$$

ANSWER: θ equals the complement of $\frac{\pi}{6}$, that is, $\theta = \frac{\pi}{3}$.

(b) $\cos\theta = \sin\frac{\pi}{4}$
ANSWER: θ equals the complement of $\frac{\pi}{4}$, that is, $\theta = \frac{\pi}{4}$.

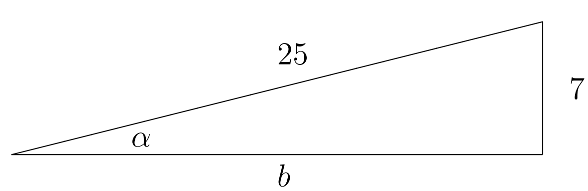
(c) $\cos\theta = \sin 25°$
ANSWER: θ equals the complement of 25°, that is, $\theta = 65°$.

4. Suppose that $\sin\alpha = \frac{7}{25}$ and $\sin\beta = -\frac{8}{17}$, with α in quadrant II and β in quadrant III.

(a) Find the exact value of $\sin(\alpha+\beta)$

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\\ &= \left(\frac{7}{25}\right)(\cos\beta)+(\cos\alpha)\left(-\frac{8}{17}\right)\end{aligned}$$

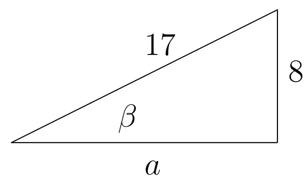
- Find $\cos\alpha$, given that α lies in quadrant II and $\sin\alpha = \frac{7}{25}$.



$\sin\alpha = \frac{\text{opp}}{\text{hyp}} = \frac{7}{25}$

Use the Pythagorean Theorem to find b.
$7^2 + b^2 = 25^2$
$49 + b^2 = 625$
$b^2 = 476$
$b = 24$
$\cos\alpha = \dfrac{\text{adj}}{\text{hyp}} = -\dfrac{24}{25}$ ←Negative because α lies in QII.

- Find $\cos\beta$ given that β lies in quadrant III and $\sin\beta = -\frac{8}{17}$.



$\sin\beta = \frac{\text{opp}}{\text{hyp}} = \frac{8}{17}$

Use the Pythagorean Theorem to find a.
$8^2 + a^2 = 17^2$
$64 + a^2 = 289$
$a^2 = 225$
$a = 15$
$\cos\beta = \dfrac{\text{adj}}{\text{hyp}} = -\dfrac{15}{17}$ ←Negative because β lies in QIII.

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \left(\frac{7}{25}\right)\left(-\frac{15}{17}\right) + \left(-\frac{24}{25}\right)\left(-\frac{8}{17}\right) = \frac{87}{425}\end{aligned}$$

(b) Find the exact value of $\cos(\alpha+\beta)$

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \left(-\frac{24}{25}\right)\left(-\frac{15}{17}\right) - \left(\frac{7}{25}\right)\left(-\frac{8}{17}\right) = \frac{416}{425}\end{aligned}$$

(c) In which quadrant does the angle $\alpha + \beta$ lie?

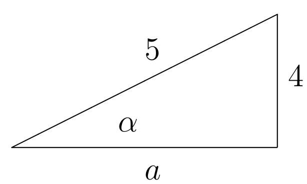
$\cos(\alpha+\beta)$ is positive, $\sin(\alpha+\beta)$ is positive, therefore, the angle $\alpha+\beta$ lies in quadrant I.

5. Suppose that $\sin\alpha = -\frac{4}{5}$ and $\cos\beta = \frac{12}{13}$, with α in quadrant III, and β in quadrant IV.

(a) Find the exact value of $\sin(\alpha+\beta)$

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \left(-\frac{4}{5}\right)\left(\frac{12}{13}\right) + \cos\alpha\sin\beta\end{aligned}$$

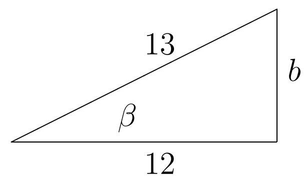
- Find $\cos\alpha$, given that α lies in quadrant III and $\sin\alpha = -\frac{4}{5}$.



$\sin\alpha = \frac{\text{opp}}{\text{hyp}} = -\frac{4}{5}$

Use the Pythagorean Theorem to find b.
$4^2 + b^2 = 5^2$
$16 + b^2 = 25$
$b^2 = 9$
$b = 3$
$\cos\alpha = \dfrac{\text{adj}}{\text{hyp}} = -\dfrac{3}{5}$ ←Negative because α lies in QII.

- Find $\sin\beta$ given that β lies in quadrant IV and $\cos\beta = \frac{12}{13}$.



$\cos\beta = \frac{\text{adj}}{\text{hyp}} = \frac{12}{13}$

Use the Pythagorean Theorem to find b.
$12^2 + b^2 = 13^2$
$144 + b^2 = 169$
$b^2 = 25$
$b = 5$
$\sin\beta = \frac{\text{opp}}{\text{hyp}} = -\frac{5}{13}$ ←Negative because β lies in QIV.

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \left(-\frac{4}{5}\right)\left(\frac{12}{13}\right) + \left(-\frac{3}{5}\right)\left(-\frac{5}{13}\right) = -\frac{33}{65}\end{aligned}$$

(b) Find the exact value of $\cos(\alpha+\beta)$

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \left(-\frac{3}{5}\right)\left(\frac{12}{13}\right) - \left(-\frac{4}{5}\right)\left(-\frac{5}{13}\right) = -\frac{56}{65}\end{aligned}$$

(c) In which quadrant does the angle $\alpha+\beta$ lie?

$\cos(\alpha+\beta)$ is negative, $\sin(\alpha+\beta)$ is negative, therefore, the angle $\alpha+\beta$ lies in quadrant III.