6.2 Verifying Trigonometric Identities

Verify that each trigonometric equation is an identity.

1. $\dfrac{\cot\theta}{\csc\theta} = \cos\theta$

   SOLUTION

LHS	RHS
$\dfrac{\cot\theta}{\csc\theta}$	$\cos\theta \quad \checkmark$
$= \dfrac{\cot\theta}{1} \cdot \dfrac{1}{\csc\theta}$	
$= \dfrac{\cos\theta}{\sin\theta} \cdot \dfrac{\sin\theta}{1}$	
$= \cos\theta \quad \checkmark$	

2. $\cos^2 x \tan^2 x = \sin^2 x$

   SOLUTION

LHS	RHS
$\cos^2 x \tan^2 x$	$\sin^2 x \quad \checkmark$
$= \dfrac{\cos^2 x}{1} \cdot \dfrac{\sin^2 x}{\cos^2 x}$	
$= \sin^2 x \quad \checkmark$	

3. $\dfrac{1-\sin^2\beta}{\cos\beta} = \cos\beta$

SOLUTION

LHS	RHS
$= \dfrac{1-\sin^2\beta}{\cos\beta}$	$\cos\beta$ ✓
$= \dfrac{\cos^2\beta}{\cos\beta}$	
$= \cos\beta$ ✓	

Aside: $\sin^2\beta + \cos^2\beta = 1 \quad \Rightarrow \quad \cos^2\beta = 1 - \sin^2\beta.$

4. $\dfrac{\tan^2\alpha + 1}{\sec\alpha} = \sec\alpha$

SOLUTION

LHS	RHS
$\dfrac{\tan^2\alpha + 1}{\sec\alpha}$	$\sec\alpha$ ✓
$= \dfrac{\sec^2\alpha}{\sec\alpha}$	
$= \sec\alpha$ ✓	

5. $1 - \csc x \sin^3 x = \cos^2 x$

SOLUTION

LHS	RHS
$1 - \csc x \sin^3 x$	$\cos^2 x \quad \checkmark$
$= 1 - \dfrac{1}{\sin x} \cdot \sin^3 x$	
$= 1 - \sin^2 x$	
$= \cos^2 x \quad \checkmark$	

6. $\cos^2 \theta(\tan^2 \theta + 1) = 1$

SOLUTION

LHS	RHS
$\cos^2 \theta(\tan^2 \theta + 1)$	$1 \quad \checkmark$
$= \cos^2 \theta(\sec^2 \theta)$	
$= \cos^2 \theta \left(\dfrac{1}{\cos^2 \theta}\right)$	
$= 1 \quad \checkmark$	

7. $\sin^2\beta(1+\cot^2\beta)=1$

SOLUTION

LHS	RHS
$\sin^2\beta(1+\cot^2\beta)$	$1 \quad \checkmark$
$=\sin^2\beta(\csc^2\beta)$	
$=\sin^2\beta\left(\dfrac{1}{\sin^2\beta}\right)$	
$=1 \quad \checkmark$	

8. $\cot\theta+\tan\theta=\sec\theta\csc\theta$

LHS	RHS
$\cot\theta+\tan\theta$	$\sec\theta\csc\theta \quad \checkmark$
$=\dfrac{\cos\theta}{\sin\theta}+\dfrac{\sin\theta}{\cos\theta}$	
$=\dfrac{\cos\theta}{\sin\theta}\cdot\dfrac{\cos\theta}{\cos\theta}+\dfrac{\sin\theta}{\cos\theta}\cdot\dfrac{\sin\theta}{\sin\theta}$	
$=\dfrac{\cos^2\theta}{\sin\theta\cos\theta}+\dfrac{\sin^2\theta}{\cos\theta\sin\theta}$	
$=\dfrac{\cos^2\theta+\sin^2\theta}{\cos\theta\sin\theta}$	
$=\dfrac{1}{\cos\theta\sin\theta}$	
$=\dfrac{1}{\cos\theta}\cdot\dfrac{1}{\sin\theta}$	
$=\sec\theta\csc\theta \quad \checkmark$	

9. $\sin^2\alpha + \tan^2\alpha + \cos^2\alpha = \sec^2\alpha$

LHS	RHS
$\sin^2\alpha + \tan^2\alpha + \cos^2\alpha$ $= (\sin^2\alpha + \cos^2\alpha) + \tan^2\alpha$ $= 1 + \tan^2\alpha$ $= \sec^2\alpha \quad \checkmark$	$\sec^2\alpha \quad \checkmark$

10. $1 + \csc^2 x \cos^2 x = \csc^2 x$

LHS	RHS
$1 + \csc^2 x \cos^2 x$ $= 1 + \dfrac{1}{\sin^2 x} x \dfrac{\cos^2 x}{1}$ $= 1 + \dfrac{\cos^2 x}{\sin^2 x}$ $= 1 + \cot^2 x$ $= \csc^2 x \quad \checkmark$	$\csc^2 x \quad \checkmark$

11. $\dfrac{\cos\alpha}{\sec\alpha} + \dfrac{\sin\alpha}{\csc\alpha} = \sec^2\alpha - \tan^2\alpha$

LHS	RHS
$\dfrac{\cos\alpha}{\sec\alpha} + \dfrac{\sin\alpha}{\csc\alpha}$	$\sec^2\alpha - \tan^2\alpha$
$== \cos\alpha\dfrac{1}{\sec\alpha} + \sin\alpha\dfrac{1}{\csc\alpha}$	$(1+\tan^2\alpha) - \tan^2\alpha$
$= \cos^2\alpha + \sin^2\alpha$	1 ✓
1 ✓	

We used $\tan^2\alpha + 1 = \sec^2\alpha$

12. $\dfrac{\sin^2\theta}{\cos\theta} = \sec\theta - \cos\theta$

LHS	RHS
$\dfrac{\sin^2\theta}{\cos\theta}$	$\sec\theta - \cos\theta$ ✓
$= \dfrac{1-\cos^2\theta}{\cos\theta}$	
$= \dfrac{1}{\cos\theta} - \dfrac{\cos^2\theta}{\cos\theta}$	
$= \sec\theta - \cos\theta$ ✓	

13. $\sin^4\theta - \cos^4\theta = 2\sin^2\theta - 1$

LHS	RHS
$\sin^4\theta - \cos^4\theta$	$2\sin^2\theta - 1 \quad \checkmark$
$= (\sin^2\theta)^2 - (\cos^2\theta)^2$	
$= (\sin^2\theta - \cos^2\theta)(\sin^2\theta + \cos^2\theta)$	
$= (\sin^2\theta - \cos^2\theta)(1)$	
$= \sin^2\theta - (1 - \sin^2\theta)$	
$= 2\sin^2\theta - 1 \quad \checkmark$	

14. $\dfrac{\cos x \sin^2 x + \cos^3 x}{\sin x} = \cot x$

LHS	RHS
$\dfrac{\cos x \sin^2 x + \cos^3 x}{\sin x}$	$\cot x \quad \checkmark$
$= \dfrac{\cos x(\sin^2 x + \cos^2 x)}{\sin x}$	
$= \dfrac{\cos x(1)}{\sin x}$	
$= \cot x \quad \checkmark$	

15. $\dfrac{\cos\theta}{\sin\theta\cot\theta} = 1$

LHS	RHS
$\dfrac{\cos\theta}{\sin\theta\cot\theta}$	1 ✓
$= \dfrac{\cos\theta}{\sin\theta\left(\frac{\cos\theta}{\sin\theta}\right)}$	
$= \dfrac{\cos\theta}{\cos\theta}$	
$= 1$ ✓	

16. $\dfrac{1-\cos\alpha}{1+\cos\alpha} = (\cot\alpha - \csc\alpha)^2$

LHS	RHS
$\dfrac{1-\cos\alpha}{1+\cos\alpha}$	$(\cot\alpha - \csc\alpha)^2$ ✓
$= \left(\dfrac{1-\cos\alpha}{1+\cos\alpha}\right)\cdot\left(\dfrac{1-\cos\alpha}{1-\cos\alpha}\right)$	
$= \dfrac{(1-\cos\alpha)^2}{1-\cos^2\alpha}$	
$= \dfrac{(1-\cos\alpha)^2}{\sin^2\alpha}$	
$= \left(\dfrac{1-\cos\alpha}{\sin\alpha}\right)^2$	
$= \left(\dfrac{1}{\sin\alpha} - \dfrac{\cos\alpha}{\sin\alpha}\right)^2$	
$= (\csc\alpha - \cot\alpha)^2$	
$= (\cot\alpha - \csc\alpha)^2$ ✓	

17. $\sin^2\theta(1+\cot^2\theta)-1=0$

LHS	RHS
$\sin^2\theta(1+\cot^2\theta)-1$	0 ✓
$=\sin^2\theta(\csc^2\theta)-1$	
$=\sin^2\theta\dfrac{1}{\sin^2\theta}-1$	
$=1-1$	
$=0$ ✓	

18. $\dfrac{\cos\theta+1}{\tan^2\theta}=\dfrac{\cos\theta}{\sec\theta-1}$

LHS	RHS
$\dfrac{\cos\theta+1}{\tan^2\theta}$ ✓	$\dfrac{\cos\theta}{\sec\theta-1}$
	$=\dfrac{\cos\theta}{\sec\theta-1}\cdot\left(\dfrac{\sec\theta+1}{\sec\theta+1}\right)$
	$=\dfrac{\cos\theta(\sec\theta+1)}{\sec^2\theta-1}$
	$=\dfrac{\cos\theta\sec\theta+\cos\theta)}{\tan^2\theta}$
	$=\dfrac{\cos\theta\left(\frac{1}{\cos\theta}\right)+\cos\theta)}{\tan^2\theta}$
	$=\dfrac{1+\cos\theta}{\tan^2\theta}$
	$=\dfrac{\cos\theta+1}{\tan^2\theta}$ ✓

19. $1 - \csc x \sin^3 x = \cos^2 x$

LHS	RHS
$1 - \csc x \sin^3 x$	$\cos^2 x \quad \checkmark$
$= 1 - \dfrac{1}{\sin x} \cdot \sin^3 x$	
$= 1 - \sin^2 x$	
$= \cos^2 x \quad \checkmark$	

20. $\dfrac{(\sec\theta - \tan\theta)^2 + 1}{\sec\theta\csc\theta - \tan\theta\csc\theta} = 2\tan\theta$

LHS	RHS
$\dfrac{(\sec\theta - \tan\theta)^2 + 1}{\sec\theta\csc\theta - \tan\theta\csc\theta}$	$2\tan\theta \quad \checkmark$
$= \dfrac{(\sec^2\theta - 2\sec\theta\tan\theta + \tan^2\theta) + 1}{\csc\theta(\sec\theta - \tan\theta)}$	
$= \dfrac{\sec^2\theta - 2\sec\theta\tan\theta + (\tan^2\theta + 1)}{\csc\theta(\sec\theta - \tan\theta)}$	
$= \dfrac{\sec^2\theta - 2\sec\theta\tan\theta + \sec^2\theta}{\csc\theta(\sec\theta - \tan\theta)}$	
$= \dfrac{2\sec^2\theta - 2\sec\theta\tan\theta}{\csc\theta(\sec\theta - \tan\theta)}$	
$= \dfrac{2\sec\theta(\sec\theta - \tan\theta)}{\csc\theta(\sec\theta - \tan\theta)}$	
$= \dfrac{2\sec\theta}{\csc\theta}$	
$= 2\sec\theta\dfrac{1}{\csc\theta}$	
$= 2\dfrac{1}{\cos\theta}\sin\theta$	
$= 2\tan\theta \quad \checkmark$	

21. $\dfrac{1}{1-\sin\theta}+\dfrac{1}{1+\sin\theta}=2\sec^2\theta$

LHS	RHS
$\dfrac{1}{1-\sin\theta}+\dfrac{1}{1+\sin\theta}$	$2\sec^2\theta$ ✓
$=\dfrac{1}{1-\sin\theta}\left(\dfrac{1+\sin\theta}{1+\sin\theta}\right)+\dfrac{1}{1+\sin\theta}\left(\dfrac{1-\sin\theta}{1-\sin\theta}\right)$	
$=\dfrac{1+\sin\theta}{1-\sin^2\theta}+\dfrac{1-\sin\theta}{1-\sin^2\theta}$	
$=\dfrac{1+\sin\theta}{\cos^2\theta}+\dfrac{1-\sin\theta}{\cos^2\theta}$	
$=\dfrac{(1+\sin\theta)+(1-\sin\theta)}{\cos^2\theta}$	
$=\dfrac{2}{\cos^2\theta}$	
$=2\sec^2\theta$ ✓	

22. $\tan^4 z = \dfrac{\sec^2 z - \csc^2 z + \csc^2 z \cos^2 z}{\cot^2 z}$

LHS	RHS
$\tan^4 z$ ✓	$\dfrac{\sec^2 z - \csc^2 z + \csc^2 z \cos^2 z}{\cot^2 z}$
	$= \tan^2 z \left(\sec^2 z - \csc^2 z + \csc^2 z \cos^2 z\right)$
	$= \tan^2 z \left(\sec^2 z - \csc^2 z(1 - \cos^2 z)\right)$
	$= \tan^2 z \left(\sec^2 z - \csc^2 z(\sin^2 z)\right)$
	$= \tan^2 z \left(\sec^2 z - \left(\frac{1}{\sin^2 z}\right) \sin^2 z\right)$
	$= \tan^2 z \left(\sec^2 z - 1\right)$
	$= \tan^2 z \left(\tan^2 z\right)$
	$= \tan^4 z$ ✓

23. $\dfrac{1}{\sec\beta - \tan\beta} = \sec\beta + \tan\beta$

LHS	RHS
$\dfrac{1}{\sec\beta - \tan\beta}$	$\sec\beta + \tan\beta$ ✓
$= \dfrac{1}{\sec\beta - \tan\beta}\left(\dfrac{\sec\beta + \tan\beta}{\sec\beta + \tan\beta}\right)$	
$= \dfrac{\sec\beta + \tan\beta}{\sec^2\beta - \tan^2\beta}$	
$= \dfrac{\sec\beta + \tan\beta}{1}$	
$= \sec\beta + \tan\beta$ ✓	

24. $\dfrac{\cot\theta + 1}{\cot\theta - 1} = \dfrac{1 + \tan\theta}{1 - \tan\theta}$

LHS	RHS
$\dfrac{\cot\theta + 1}{\cot\theta - 1}$	$\dfrac{1 + \tan\theta}{1 - \tan\theta}$
$= \dfrac{\frac{\cos\theta}{\sin\theta} + 1}{\frac{\cos\theta}{\sin\theta} - 1}$	$= \dfrac{1 + \frac{\sin\theta}{\cos\theta}}{1 - \frac{\sin\theta}{\cos\theta}}$
$= \dfrac{\frac{\cos\theta}{\sin\theta} + 1}{\frac{\cos\theta}{\sin\theta} - 1} \cdot \left(\dfrac{\sin\theta}{\sin\theta}\right)$	$= \dfrac{1 + \frac{\sin\theta}{\cos\theta}}{1 - \frac{\sin\theta}{\cos\theta}} \cdot \left(\dfrac{\cos\theta}{\cos\theta}\right)$
$= \dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$ ✓	$= \dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$ ✓

25. $\sin^3 x \csc x + \cos^3 x \sec x = 1$

LHS	RHS
$\sin^3 x \csc x + \cos^3 x \sec x$	1 ✓
$= \sin^3 x \dfrac{1}{\sin x} + \cos^3 x \dfrac{1}{\cos x}$	
$= \sin^2 x + \cos^2 x$	
$= 1$ ✓	

26. $\dfrac{\csc\theta + \cot\theta}{\tan\theta + \sin\theta} = \cot\theta\csc\theta$

LHS	RHS
$\dfrac{\csc\theta + \cot\theta}{\tan\theta + \sin\theta}$	$\cot\theta\csc\theta$ ✓
$= (\csc\theta + \cot\theta)\left(\dfrac{1}{\tan\theta + \sin\theta}\right)$	
$= \left(\dfrac{1}{\sin\theta} + \dfrac{\cos\theta}{\sin\theta}\right)\left(\dfrac{1}{\frac{\sin\theta}{\cos\theta} + \sin\theta}\right)$	
$= \left(\dfrac{1+\cos\theta}{\sin\theta}\right)\left(\dfrac{1}{\frac{\sin\theta}{\cos\theta} + \sin\theta}\right)\dfrac{\cos\theta}{\cos\theta}$	
$= \left(\dfrac{1+\cos\theta}{\sin\theta}\right)\left(\dfrac{\cos\theta}{\sin\theta + \sin\theta\cos\theta}\right)$	
$= \left(\dfrac{1+\cos\theta}{\sin\theta}\right)\left(\dfrac{\cos\theta}{\sin\theta(1++\cos\theta\theta}\right)$	
$= \dfrac{1}{\sin\theta}\cdot\dfrac{\cos\theta}{\sin\theta}$	
$= \csc\theta\cot\theta$	
$= \cot\theta\csc\theta$ ✓	