

6.1 Fundamental Identities

Fundamental Identities

An **identity** is an equation that is satisfied by every value in the domain of its variable.

- Reciprocal Identities. The co's don't go.

$$\cot \theta = \frac{1}{\tan \theta} \quad \sec \theta = \frac{1}{\cos \theta} \quad \csc \theta = \frac{1}{\sin \theta}$$

- Quotient Identities.

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \cot \theta = \frac{\cos \theta}{\sin \theta}$$

- Pythagorean Identities.

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

- Other Forms of the Pythagorean Identities

$$\sin^2 \theta = 1 - \cos^2 \theta$$

$$\cos^2 \theta = 1 - \sin^2 \theta$$

- Odd and Even Identities

Odd Identities

$$\sin(-\theta) = -\sin \theta$$

$$\csc(-\theta) = -\csc \theta$$

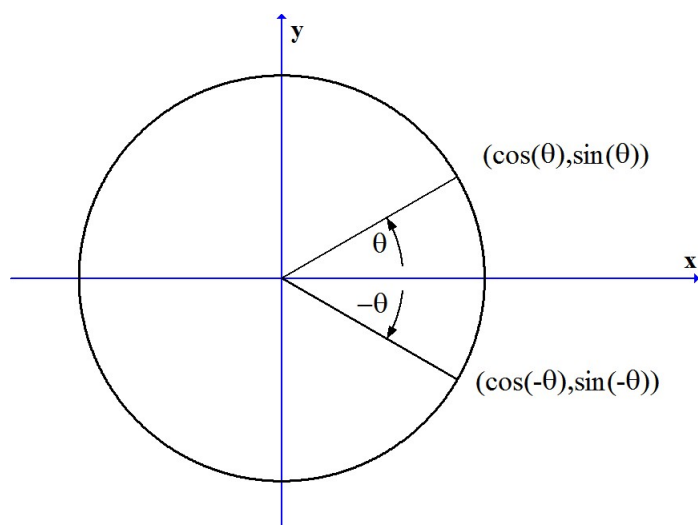
$$\tan(-\theta) = -\tan \theta$$

$$\cot(-\theta) = \cot \theta$$

Even Identities

$$\cos(-\theta) = \cos \theta$$

$$\sec(-\theta) = \sec \theta$$



6.2 Verifying Trigonometric Identities

In this section we will prove the validity trigonometric identities. We will assume that the identities given in the previous section are true.

Example Verify that the following equation is an identity.

$$\cot \theta + 1 = \csc \theta (\cos \theta + \sin \theta)$$

SOLUTION

LHS	RHS
$\cot \theta + 1$	$\csc \theta (\cos \theta + \sin \theta)$
	$= \frac{1}{\sin \theta} (\cos \theta + \sin \theta)$
	$= \frac{\cos \theta}{\sin \theta} + \frac{\sin \theta}{\sin \theta}$
	$= \cot \theta + 1$
	$= \text{LHS}$

Example Verify that the following equation is an identity.

$$\tan^2 x(1 + \cot^2 x) = \frac{1}{1 - \sin^2 x}$$

SOLUTION

LHS	RHS
$\tan^2 x(1 + \cot^2 x)$	$\frac{1}{1 - \sin^2 x}$
$= \tan^2 x + \tan^2 x \cot^2 x$	$= \frac{1}{\cos^2 x}$
$= \tan^2 x + \frac{\sin^2 x}{\cos^2 x} \cdot \frac{\cos^2 x}{\sin^2 x}$	$= \sec^2 x$
$= \tan^2 x + 1$	$= \sec^2 x$
$= \sec^2 x \quad \checkmark$	$= \sec^2 x \quad \checkmark$

Example Verify that the following equation is an identity.

$$\frac{\tan t - \cot t}{\sin t \cos t} = \sec^2 t - \csc^2 t$$

SOLUTION

LHS	RHS
$\frac{\tan t - \cot t}{\sin t \cos t}$	$\sec^2 t - \csc^2 t$
$(\tan t - \cot t) \frac{1}{\sin t \cos t}$	
$\left(\frac{\sin t}{\cos t} - \frac{\cos t}{\sin t}\right) \frac{1}{\sin t \cos t}$	
$\frac{\sin t}{\cos t} \cdot \frac{1}{\sin t \cos t} - \frac{\cos t}{\sin t} \cdot \frac{1}{\sin t \cos t}$	
$\frac{1}{\cos^2 t} - \frac{1}{\sin^2 t}$	
$\sec^2 t - \csc^2 t$	
$= \text{RHS}$	

Example Verify that the following equation is an identity.

$$\frac{\cos x}{1 - \sin x} = \frac{1 + \sin x}{\cos x}$$

SOLUTION

LHS	RHS
$\frac{\cos x}{1 - \sin x}$	$\frac{1 + \sin x}{\cos x}$
$\frac{\cos x}{1 - \sin x} \cdot \frac{1 + \sin x}{1 + \sin x}$	
$\frac{\cos x(1 + \sin x)}{1 - \sin^2 x}$	
$\frac{\cos x(1 + \sin x)}{\cos^2 x}$	
$\frac{1 + \sin x}{\cos x}$	
= RHS	

Example Verify that the following equation is an identity.

$$\frac{\sec \alpha + \tan \alpha}{\sec \alpha - \tan \alpha} = \frac{1 + 2 \sin \alpha + \sin^2 \alpha}{\cos^2 \alpha}$$

SOLUTION

LHS	RHS
$\frac{\sec \alpha + \tan \alpha}{\sec \alpha - \tan \alpha}$	$\frac{1 + 2 \sin \alpha + \sin^2 \alpha}{\cos^2 \alpha}$
$\frac{\sec \alpha + \tan \alpha}{\sec \alpha - \tan \alpha} \cdot \left(\frac{\sec \alpha + \tan \alpha}{\sec \alpha + \tan \alpha} \right)$	$\frac{1}{\cos^2 \alpha} + 2 \frac{\sin \alpha}{\cos^2 \alpha} + \frac{\sin^2 \alpha}{\cos^2 \alpha}$
$\frac{(\sec \alpha + \tan \alpha)^2}{\sec^2 \alpha - \tan^2 \alpha}$	$\sec^2 \alpha + 2 \frac{1}{\cos \alpha} \cdot \frac{\sin \alpha}{\cos \alpha} + \tan^2 \alpha$
$\frac{(\sec \alpha + \tan \alpha)^2}{1}$	$\sec^2 \alpha + 2 \sec \alpha \tan \alpha + \tan^2 \alpha \quad \checkmark$
$\sec^2 \alpha + 2 \sec \alpha \tan \alpha + \tan^2 \alpha \quad \checkmark$	

More Examples

Example Verify that the following equation is an identity.

1. $1 + \sec x \sin x \tan x = \sec^2 x$

SOLUTION

LHS	RHS
$1 + \sec x \sin x \tan x$	$\sec^2 x \quad \checkmark$
$1 + \frac{1}{\cos x} \cdot \frac{\sin x}{1} \cdot \frac{\sin x}{\cos x}$	
$1 + \frac{\sin^2 x}{\cos^2 x}$	
$1 + \tan^2 x$	
$\sec^2 x \quad \checkmark$	

$$2. \frac{\cos \alpha}{1 - \sin \alpha} = \frac{1 + \sin \alpha}{\cos \alpha}$$

SOLUTION

LHS	RHS
$\frac{\cos \alpha}{1 - \sin \alpha}$	$\frac{1 + \sin \alpha}{\cos \alpha} \quad \checkmark$
$\frac{\cos \alpha}{1 - \sin \alpha} \cdot \left(\frac{1 + \sin \alpha}{1 + \sin \alpha} \right)$	
$\frac{\cos \alpha(1 + \sin \alpha)}{1 - \sin^2 \alpha}$	
$\frac{\cos \alpha(1 + \sin \alpha)}{\cos^2 \alpha}$	
$\frac{1 + \sin \alpha}{\cos \alpha} \quad \checkmark$	

$$3. \frac{\csc x - \sin x}{\sin x} = \cot^2 x$$

SOLUTION

LHS	RHS
$\frac{\csc x - \sin x}{\sin x}$	$\cot^2 x \quad \checkmark$
$(\csc x - \sin x) \frac{1}{\sin x}$	
$\left(\frac{1}{\sin x} - \frac{\sin x}{1}\right) \frac{1}{\sin x}$	
$\left(\frac{1}{\sin x} \cdot \frac{1}{\sin x}\right) - \left(\frac{\sin x}{1} \cdot \frac{1}{\sin x}\right)$	
$\frac{1}{\sin^2 x} - 1$	
$\csc^2 x - 1$	
$\cot^2 x \quad \checkmark$	

$$4. \ 2 \tan^2 x = \frac{1}{\csc x - 1} - \frac{1}{\csc x + 1}$$

SOLUTION

LHS	RHS
$2 \tan^2 x \quad \checkmark$	$\frac{1}{\csc x - 1} - \frac{1}{\csc x + 1}$ $\frac{1}{\csc x - 1} \cdot \left(\frac{\csc x + 1}{\csc x + 1} \right) - \frac{1}{\csc x + 1} \cdot \left(\frac{\csc x - 1}{\csc x - 1} \right)$ $\frac{\csc x + 1}{\csc^2 x - 1} - \frac{\csc x - 1}{\csc^2 x - 1}$ $\frac{(\csc x + 1) - (\csc x - 1)}{\csc^2 x - 1}$ $\frac{\csc x + 1 - \csc x + 1}{\csc^2 x - 1}$ $\frac{2}{\csc^2 x - 1}$ $\frac{2}{\cot^2 x}$ $2 \tan^2 x \quad \checkmark$

$$5. \frac{1 - \sin^2 t}{1 - \csc(-t)} = \frac{1 + \sin(-t)}{\csc t}$$

SOLUTION

LHS	RHS
$\frac{1 - \sin^2 t}{1 - \csc(-t)}$	$\frac{1 + \sin(-t)}{\csc t}$
$\frac{1 - \sin^2 t}{1 + \csc t}$	$\frac{1 - \sin(t)}{\csc t}$
$\frac{\cos^2 t}{1 + \csc t} \quad \checkmark$	$\frac{1 - \sin t}{\csc t} \cdot \left(\frac{1 + \sin t}{1 + \sin t}\right)$
	$\frac{1 - \sin^2 t}{(\csc t)(1 + \sin t)}$
	$\frac{\cos^2 t}{(\csc t)(1 + \sin t)}$
	$\frac{\cos^2 t}{\csc t + \csc t \sin t}$
	$\frac{\cos^2 t}{\csc t + \left(\frac{1}{\sin t}\right) \left(\frac{\sin t}{1}\right)}$
	$\frac{\cos^2 t}{\csc t + 1}$
	$\frac{\cos^2 t}{1 + \csc t} \quad \checkmark$