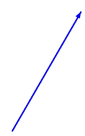
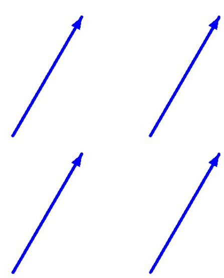
7.3 Vectors

A **vector** in the plane is a directed line segment. It has a starting point and an ending point. We can draw it as an arrow.



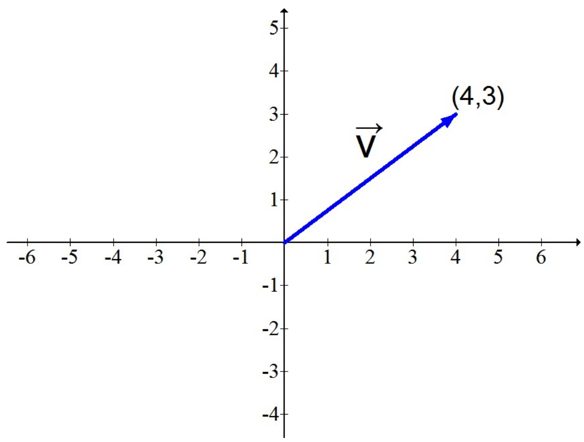
Two properties uniquely describe a vector: **magnitude** and **direction**. The magnitude is the vector's length. Vectors are used to describe velocity because velocity has direction (which way an object is going), and velocity has magnitude (the speed of the object). Vectors are also be used to describe acceleration and force.

We can place a vector anywhere in the plane. Below the same vector is shown with four different starting points.



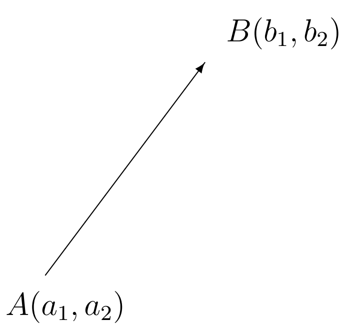
We can distinguish the vectors shown above by taking into account their starting points. If we take into account a vector's starting point, then it is called a **located vector**. When used in applications, it is pretty common for the word vector to be used when what is really meant is located vector. Physicists and applied mathematicians are not as fussy as pure mathematicians.

If a vector is placed in the plane so that its starting point is the origin, then the vector can be uniquely described by the coordinates of its endpoint. The vector **v** shown below with starting point at the origin can be denoted by its endpoint $(4, 3)$. We write $\mathbf{v} = \langle 4, 3\rangle$. This is called the **component form** of the vector **v**. Even if **v** is placed so that its starting point is not the origin, we still can call the vector $\mathbf{v} = \langle 4, 3\rangle$.



It follows that the set of vectors in the plane can be identified with the points in the plane. The points in the plane can in turn be identified with the set of all ordered pairs of real numbers, denoted $\mathbb{R}^2$. We define $\mathbb{R}^2 = \{(x, y) |\ x, y \text{ are real numbers}\}$.

The vector with starting point $A(a_1, a_2)$ and ending point $B(b_1, b_2)$ is denoted $\overrightarrow{AB}$.



To write $\overrightarrow{AB}$ in component form, we translate the vector so that its starting point is the origin. We shift horizontally by a_1 and vertically by a_2. The endpoint of the translated vector is then $(b_1 - a_1, b_2 - a_2)$.

The component form of the vector $\overrightarrow{AB}$ where $A = (a_1, a_2)$ and $B = (b_1, b_2)$ is

$$\overrightarrow{AB} = \langle a_2 - a_1, b_2 - b_1 \rangle$$

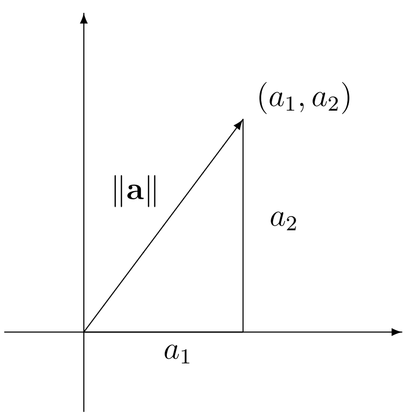
Example Find the component form of the vector starting at $A(2, 3)$ and ending at $B(-7, 5)$.

SOLUTION $\overrightarrow{AB} = \langle 5 - 3, -7 - 2 \rangle = \langle 2, -9 \rangle$.

The representation of a vectors as an ordered pairs is useful because we can do algebra with the vectors. On the other hand, it is sometimes useful to go back to the representation of a vector as a directed line segment and draw a picture.

The Magnitude or Length of a Vector

We can find the length of a vector $\mathbf{a} = \langle a_1, a_2 \rangle$ by using the Pythagorean Theorem.



The length of $\mathbf{a}$ is denoted $\|\mathbf{v}\|$. By the Pythagorean Theorem, we have

$$\|\mathbf{a}\|^2 = a_1^2 + a_2^2.$$

We get

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2}.$$

Example Find the length of the given vectors.

1. $\mathbf{a} = \langle 3, 4\rangle$

   SOLUTION
   $$\|\mathbf{a}\| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

2. $\mathbf{b} = \langle 5, 12\rangle$

   SOLUTION
   $$\|\mathbf{b}\| = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$$

3. $\mathbf{c} = \langle 1, 3\rangle$

   SOLUTION
   $$\|\mathbf{b}\| = \sqrt{1^2 + 3^2} = \sqrt{10}$$

Scalar Multiplication

When working with vectors in the plane, a **scalar** is a real number. Let $\mathbf{a} = \langle a_1, a_2\rangle$ be a vector and let c be a scalar. The we define the vector $c\mathbf{v}$ as

$$c\mathbf{a} = c\langle a_1, a_2\rangle = \langle ca_1, ca_2\rangle.$$

Example Multiply.

1. $2\langle 3, 1\rangle$

   SOLUTION $2\langle 3, 1\rangle = \langle 2(3), 2(1)\rangle = \langle 6, 2\rangle$

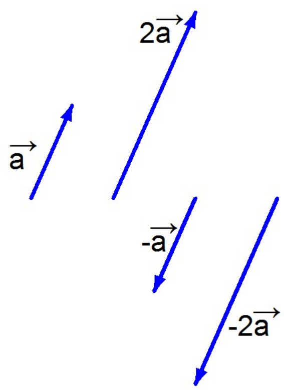
2. $-4\langle 3, 1\rangle$

   SOLUTION $-4\langle 3, 1\rangle = \langle -4(3), -4(1)\rangle = \langle -12, -4\rangle$

Example Let $\mathbf{v} = \langle 1, 2\rangle$. Find the vectors $2\mathbf{v}$, $(-1)\mathbf{v}$, and $-2\mathbf{v}$. Sketch the vectors.

SOLUTION

- $2\mathbf{v} = 2\langle 1, 2\rangle = \langle 2, 4\rangle$
- $(-1)\mathbf{v} = (-1)\langle 1, 2\rangle = \langle -1, -2\rangle$
- $-2\mathbf{v} = (-2)\langle 1, 2\rangle = \langle -2, -4\rangle$



We can see that $2\mathbf{a}$ is twice the length of $\mathbf{a}$. We can also see that $-\mathbf{a}$ and $-2\mathbf{a}$ have opposite direction as $\mathbf{a}$. When a vector $\mathbf{v}$ is multiplied by scalar c, and $c \neq 1$, its length is changed. If the scalar c is positive, then the vector $c\mathbf{v}$ has the same direction as $\mathbf{v}$. If the scalar c is negative, then then the vector $c\mathbf{v}$ has the opposite direction to $\mathbf{v}$.

We get the general result

$$\|c\mathbf{v}\| = |c|\|\mathbf{v}.\|$$

PROOF Let $\mathbf{v} = \langle v_1, v_2 \rangle$. Then

$$\begin{aligned}
\|c\mathbf{v}\| &= \|c\langle v_1, v_2\rangle\| \\
&= \|\langle cv_1, cv_2\rangle\| \\
&= \sqrt{(cv_1)^2 + (cv_2)^2} \\
&= \sqrt{c^2v_1^2 + c^2v_2^2} \\
&= \sqrt{c^2(v_1^2 + v_2^2)} = \sqrt{c^2}\sqrt{v_1^2 + v_2^2} = |c|\|\mathbf{v}\|
\end{aligned}$$

Two vectors nonzero $\mathbf{a}$ and $\mathbf{b}$ are said to be **parallel** if one is a scalar multiple of the other. That is, $\mathbf{a}$ and $\mathbf{b}$ are parallel if there is a nonzero constant c such that $\mathbf{a} = c\mathbf{b}$.

Addition and Subtraction of Vectors

Let $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ be vectors. We define the **sum** $\mathbf{a} + \mathbf{b}$ to be

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle.$$

Thus we define the sum componentwise.

Example Let $\mathbf{a} = \langle 1, 4 \rangle$ and $\mathbf{b} = \langle -1, 5 \rangle$. Then

$$\mathbf{a} + \mathbf{b} = \langle 1 - 1, 4 + 5 \rangle = \langle 0, 9 \rangle.$$

The following properties follow from the definition of vector addition. For any vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, we have the

Commutative Property. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.

Associative Property. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$.

Zero Vector. Let $\mathbf{0} = \langle 0, 0 \rangle$. Then $\mathbf{a} + \mathbf{0} = \mathbf{a}$ for any vector $\mathbf{a}$.

Additive Inverse. If $\mathbf{a} = \langle a_1, a_2 \rangle$, then the vector

$$-\mathbf{a} = \langle -a_1, -a_2 \rangle \text{ is such that } \mathbf{a} + -\mathbf{a} = \mathbf{0}.$$

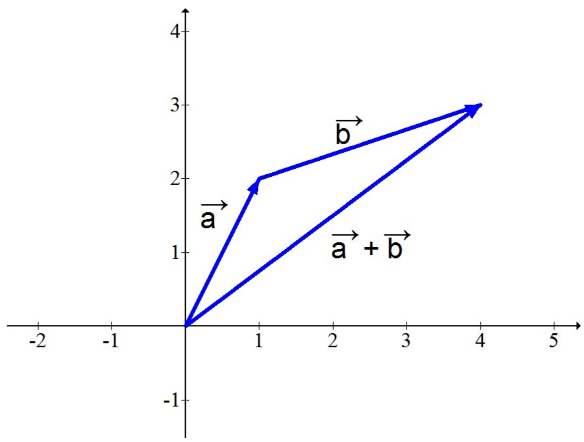
Note that $(-1)\mathbf{a} = -\mathbf{a}$

Example Let $\mathbf{a} = \langle 1, 2\rangle$ and $\mathbf{b} = \langle 3, 1\rangle$. Find $\mathbf{a} + \mathbf{b}$, then draw vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} + \mathbf{b}$.

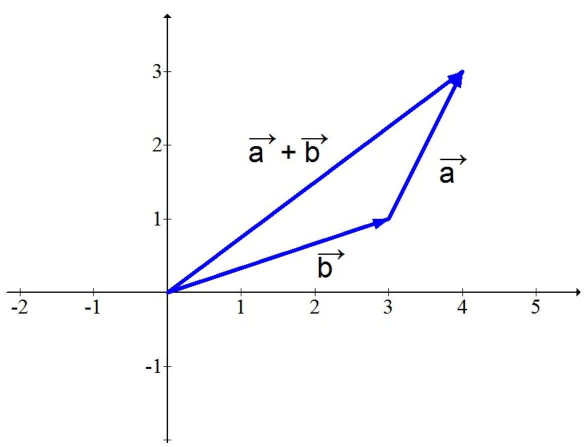
Solution By the definition of vector addition

$$\mathbf{a} + \mathbf{b} = \langle 1, 2\rangle + \langle 3, 1\rangle = \langle 4, 3\rangle.$$

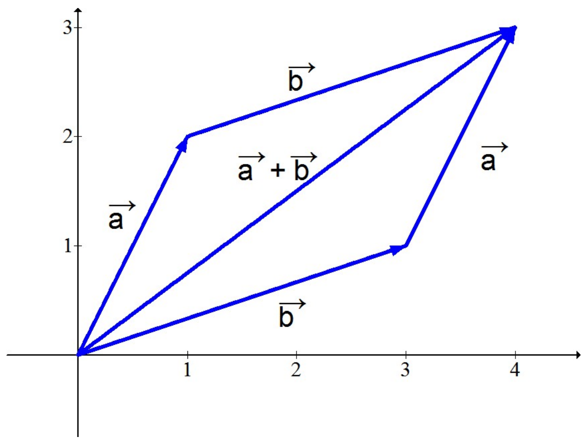
A graphical representation of this addition is shown by placing $\mathbf{b}$ so that its starting point is at the endpoint of $\mathbf{a}$ as shown in the picture below. The endpoint of $\mathbf{b}$ when it starts at the endpoint of $\mathbf{a}$ is the vector $\mathbf{a} + \mathbf{b}$.



We can also find $\mathbf{a} + \mathbf{b}$ by placing $\mathbf{a}$ so that its starting point is at the endpoint of $\mathbf{b}$. The endpoint of $\mathbf{a}$ when it starts at the endpoint of $\mathbf{b}$ is the vector $\mathbf{a} + \mathbf{b}$.



When we show both ways of getting to $\mathbf{a} + \mathbf{b}$, we get a parallelogram. This is called the **parallelogram law**.



Example Evaluate as a single vector.

1. $2\langle 2, -3\rangle + 5\langle -1, 4\rangle$

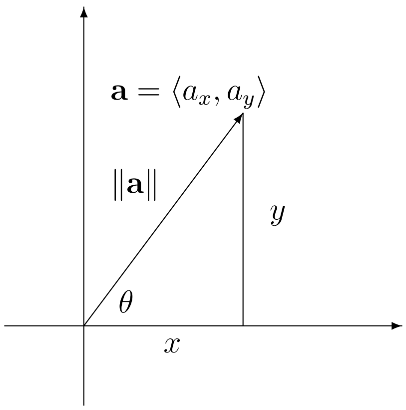
   SOLUTION $= \langle 4, -6\rangle + \langle -5, 20\rangle = \langle -1, 14\rangle$

2. $5\langle -1, 0\rangle - 3\langle 2, 3\rangle$

   SOLUTION $= \langle -5, 0\rangle + \langle -6, -9\rangle = \langle -14, -9\rangle$

Horizontal and Vertical Components, Direction Angle

Let $\mathbf{a} = \langle a_x, a_y\rangle$ be a vector. Then a_x is called the horizontal component, and a_y is called the vertical component. The angle θ that the vector $\mathbf{a}$ forms with the positive x-axis is called the **direction angle**.



By looking at the picture above, we see that

$$\cos\theta = \frac{a_x}{\|\mathbf{a}\|} \text{ and } \sin\theta = \frac{a_y}{\|\mathbf{a}\|}$$

Equivalently

$$a_x = \|\mathbf{a}\|\cos\theta \text{ and } a_y = \|\mathbf{a}\|\sin\theta$$

These are the same formulas we have when we convert from polar coordinates (r, θ) to rectangular coordinates (x, y) where r corresponds to the magnitude $\|\mathbf{a}\|$.

Any vector $\mathbf{a}$ can be written as

$$\begin{aligned} \mathbf{a} &= \langle a_x, a_y \rangle = \langle \|\mathbf{a}\|\cos\theta, \|\mathbf{a}\|\sin\theta \rangle \\ &= \|\mathbf{a}\|\langle \cos\theta, \sin\theta \rangle \end{aligned}$$

Example The vector $\mathbf{a}$ has magnitude 8 and direction angle $\theta = 120°$. Write $\mathbf{a}$ in the component form $\langle a_1, a_2 \rangle$.

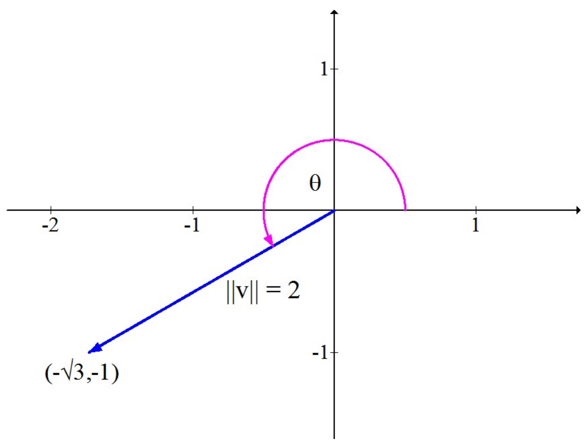
SOLUTION

$$\begin{aligned} \mathbf{a} &= \|\mathbf{a}\|\langle \cos\theta, \sin\theta \rangle \\ &= 8\langle \cos 120°, \sin 120° \rangle \\ &= 8\left\langle 1/2, \sqrt{3}/2 \right\rangle \\ &= \langle 4, 4\sqrt{3} \rangle \end{aligned}$$

Example Find the magnitude and direction angle of the vector $\mathbf{a} = \langle -\sqrt{3}, -1 \rangle$.

SOLUTION This is essentially the same problem as converting $(-\sqrt{3}, -1)$ in rectangular coordinates to (r, θ) in polar coordinates, where r corresponds to the magnitude. We find the magnitude the same way we found the radius.

$$\|\mathbf{a}\| = \sqrt{(-\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$$



If the magnitude is 2, then

$$a_x = \|\mathbf{v}\| \cos\theta \quad \text{and} \quad a_y = \|\mathbf{v}\| \sin\theta$$

$$-\sqrt{3} = 2\cos\theta \quad \text{and} \quad -1 = 2\sin\theta$$

This gives us

$$\cos\theta = -\frac{\sqrt{3}}{2} \quad \text{and} \quad \sin\theta = -\frac{1}{2}$$

We conclude that $\theta = 7\pi/6 = 210°$.

Unit Vectors

A **unit vector** is a vector with magnitude 1. Let $\mathbf{v}$ be a nonzero vector. Then the vector $\mathbf{v}/\|\mathbf{v}\|$ is a unit vector parallel to $\mathbf{v}$. It is parallel to $\mathbf{v}$

because it is a scalar multiple of $\mathbf{v}$. We can see that it is a unit vector by finding its magnitude.

$$\left\| \frac{\mathbf{v}}{\|\mathbf{v}\|} \right\| = \frac{1}{\|\mathbf{v}\|}\|\mathbf{v}\| = 1$$

Example Find a unit vector parallel to $\mathbf{v} = \langle -3, 4 \rangle$.

SOLUTION $\|\mathbf{v}\| = \sqrt{(-3)^3 + 4^2} = \sqrt{25} = 5$. The vector $\mathbf{u}$ is parallel to $\mathbf{v}$ where

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\|\mathbf{v}\|}\|\mathbf{v}\| = \frac{1}{5}\langle -3, 4 \rangle = \langle -3/5, 4/5 \rangle.$$

There are two important unit vectors $\mathbf{i}$ and $\mathbf{j}$ defined as

$$\mathbf{i} = \langle 1, 0 \rangle, \quad \mathbf{j} = \langle 0, 1 \rangle$$

The vector $\mathbf{i}$ is a unit vector parallel to the x-axis. The vector $\mathbf{j}$ is a unit vector parallel to the y-axis.

Let $\mathbf{a} = \langle a_1, a_2 \rangle$ be a vector. Then we can write $\mathbf{a}$ in terms of $\mathbf{i}$ and $\mathbf{j}$ as follows.

$$\mathbf{a} = \langle a_1, a_2 \rangle = \langle a_1, 0 \rangle + \langle 0, a_2 \rangle = a_1\langle 1, 0 \rangle + a_2\langle 0, 1 \rangle = a_1\mathbf{i} + a_2\mathbf{j}$$

Example Write the vector $\mathbf{a} = \langle -3, 4 \rangle$ as a linear combination of vectors $\mathbf{i}$ and $\mathbf{j}$.

SOLUTION $\mathbf{a} = \langle -3, 4 \rangle = -3\mathbf{i} + 4\mathbf{j}$

Example Write the vector $-2\mathbf{i} + 7\mathbf{j}$ in component form $\langle a_1, a_2 \rangle$.

SOLUTION $-2\mathbf{i} + 7\mathbf{j} = \langle -2, 7 \rangle$.

Dot Product

Let $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$. We define the **dot product** between $\mathbf{a}$ and $\mathbf{b}$ as

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2$$

Example Find the dot product.

1. $\langle -2, 3 \rangle \cdot \langle 4, 1 \rangle$

   SOLUTION $=(-2)(4) + (3)(1) = -5$

2. $\langle 2, 0 \rangle \cdot \langle -3, 5 \rangle$

   SOLUTION $=(2)(-3) + (0)(-5) = -6$

Below are a list of the properties of the dot product.

Property Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be vectors, and m a real number. Then

1. $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(ma) \cdot \mathbf{b} = m(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (m\mathbf{b})$
5. $\mathbf{0} \cdot \mathbf{a} = 0$

The Angle Between Two Vectors

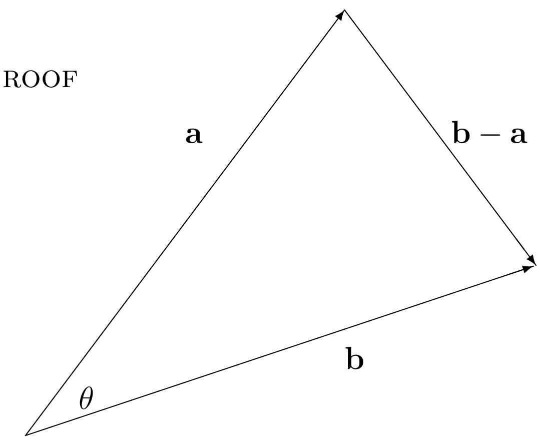
If θ is the angle between two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$$

We can equivalently write

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|}$$

PROOF



Let $\mathbf{a} = \langle a_1, a_1 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$. Then $\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$. The Law of Cosines give

$$\begin{aligned} \|\mathbf{b} - \mathbf{a}\|^2 &= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta \\ (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) &= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta \\ \mathbf{b} \cdot \mathbf{b} - 2\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{a} &= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta \\ \|\mathbf{b}\|^2 - 2\mathbf{a} \cdot \mathbf{b} + \|\mathbf{a}\|^2 &= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta \\ -2\mathbf{a} \cdot \mathbf{b} &= -2\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta \\ \cos\theta &= \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} \qquad \square \end{aligned}$$

Example Find the angle θ between the vectors $\mathbf{a} = \langle 4, -3 \rangle$ and $\mathbf{b} = \langle 5, 12 \rangle$.

SOLUTION $\|\mathbf{a} = \sqrt{4^2 + (-3)^2} = \sqrt{25} = 5$. $\|\mathbf{a} = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$. $\mathbf{a} \cdot \mathbf{b} = \langle 4, -3 \rangle \cdot \langle 5, 12 \rangle = (4)(5) + (-3)(12) = -16$.

$$\begin{aligned} \cos\theta &= \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \frac{-16}{5 \cdot 13} = -\frac{16}{65} \\ \theta &= \cos^{-1}\left(-\frac{16}{65}\right) \approx 104.25^\circ \end{aligned}$$

Let θ be the angle between vectors $\mathbf{a}$ and $\mathbf{b}$. We said before that two nonzero vectors are parallel if and only if they are scalar multiples of each other. It follows that two vectors **a** and **b** are parallel if $\theta = 0$ or $\theta = \pi$. We say that vectors **a** and **b** are **orthogonal** if $\theta = \pi/2$.

Theorem Two vectors a and b are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Example Are the vectors $\langle 1, 4\rangle$ and $\langle -8, 2\rangle$ orthogonal?

SOLUTION Yes. $\langle 1, 4\rangle \cdot \langle -8, 2\rangle = (1)(-8) + (4)(2) = 0$.

Projections

Let θ be the angle between two nonzero vectors **a** and **b**. Then the component of **a** along **b**, denoted by $\text{comp}_{\mathbf{b}}\mathbf{a}$, is given by

$$\text{comp}_{\mathbf{b}}\mathbf{a} = \|\mathbf{a}\| \cos\theta$$

Formula If a and b are nonzero vectors, then

$$\text{comp}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|}$$