Let $\mathcal{R}$ be the region bounded by the curve $y = f(x)$ and the $x$-axis, where $a \leq x \leq b$.

Next, we rotate the region $\mathcal{R}$ about the $x$-axis.

We then slice the solid into $n$ disks.
One typical disc is close in shape to a cylinder with radius \( r = f(x_i) \) and height \( h = \Delta x \). The volume of the approximating cylinder is \( V_i = \pi r^2 h = \pi (f(x_i))^2 \Delta x \).

We sum the volumes of the approximating cylinders.

\[
\sum_{i=1}^{n} \pi (f(x_i))^2 \Delta x
\]
The greater the number of disks, \( n \), the better the approximation. We take the limit as \( n \) approaches infinity.

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \pi \left( f(x_i) \right)^2 \Delta x
\]

We recognize the limit as a Riemann integral. We define the volume of a solid of rotation by

\[
\text{Volume} = \int_{a}^{b} \pi \left( f(x) \right)^2 \, dx
\]

**Example 1** Solid Obtained by Rotating a Region about the \( x \)-axis. Find the volume of the solid obtained by rotating about the \( x \)-axis the region under the curve \( y = \sqrt{x} \) from 0 to 1.

**Solution** \( \pi/2 \).

**Example 2** Solid Obtained by Rotating a Region about the \( y \)-axis. Find the volume of the solid obtained by rotating about the \( y \)-axis the region bounded by \( y = x^3 \), \( y = 8 \) and \( x = 0 \) about the \( y \)-axis.

**Solution** \( 96\pi/5 \).

**Example 3** The region \( R \) enclosed by the curves \( y = x \) and \( y = x^2 \) is rotated about the \( x \)-axis. Find the volume of the resulting solid.

**Solution** \( 2\pi/15 \).

**Example 4** The region \( R \) enclosed by the curves \( y = x \) and \( y = x^2 \) is rotated about the line \( y = 2 \). Find the volume of the resulting solid.

**Solution** \( 8\pi/15 \).

**Example 5** The region \( R \) enclosed by the curves \( y = x \) and \( y = x^2 \) is rotated about the line \( x = -1 \). Find the volume of the resulting solid.

**Solution** \( \pi/2 \).
Homework

*Disk/Washer Method:* Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region, the solid, and a typical disk or washer.

1. \( y = 1/x, \ x = 1, \ x = 2, \ y = 0; \) about the \( x \)-axis.
2. \( y = 1 - x^2, \ y = 0; \) about the \( x \)-axis.
3. \( x = 2\sqrt{y}, \ x = 0, \ y = 9; \) about the \( y \)-axis.
4. \( y = \ln x, \ y = 1, \ y = 2, \ x = 0; \) about the \( y \)-axis.
5. \( y = x^3, \ y = x, \ x \geq 0; \) about the \( x \)-axis.
6. \( y = \frac{1}{4}x^2, \ y = 5 - x^2; \) about the \( x \)-axis.
7. \( y^2 = x, \ x = 2y; \) about the \( y \)-axis.
8. \( y = x^{2/3}, \ x = 1, \ y = 0; \) about the \( y \)-axis.
6.2 Solids of Rotation: The Disk/Washer Method

Solutions:

Disk/Washer Method about the $x$-axis

\[ V = \int_a^b \left[ \pi (f(x))^2 - \pi (g(x))^2 \right] \, dx \]

Disk/Washer Method about the $y$-axis

\[ V = \int_c^d \left[ \pi (f(y))^2 - \pi (g(y))^2 \right] \, dy \]

1. $\pi/2$

2. $y = 1 - x^2$, $y = 0$; about the $x$-axis.

\[
V = \int_{-1}^{1} \pi (1 - x^2)^2 \, dx = 2\pi \int_{0}^{1} (1 - x^2)^2 \, dx = 2\pi \int_{0}^{1} (1 - 2x^2 + x^4) \, dx
\]

\[
= 2\pi \left[ x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_{0}^{1} = 2\pi \left[ 1 - \frac{2}{3} + \frac{1}{5} \right] = \frac{16\pi}{15}
\]

3. $162\pi$

4. $y = \ln x$, $y = 1$, $y = 2$, $x = 0$; about the $y$-axis.

\[ y = \ln x \iff x = e^y \]

\[
V = \int_1^2 \pi (e^y)^2 \, dy = \pi \int_1^2 e^{2y} \, dy
\]

Use $u$-substitution: $u = 2y$, $du = 2dy$, $\frac{1}{2}du = dy$

We have $u = 2$ when $x = 1$, and $u = 4$ when $x = 2$.

\[
= \frac{\pi}{2} \int_2^4 e^u \, du = \frac{\pi}{2} \left[ e^u \right]_2^4 = \frac{\pi}{2} \left( e^4 - e^2 \right)
\]
5. \(4\pi/21\)

6. \(y = \frac{x^2}{4}, \ y = 5 - x^2\); about the \(x\)-axis.

Points of Intersection:
\[
\frac{x^2}{4} = 5 - x^2 \\
x^2 = 20 - 4x^2 \\
5x^2 = 20 \\
x = \pm 2
\]

\[
V = \int_{-2}^{2} \left( \pi (5 - x^2)^2 - \pi \left( \frac{x^2}{4} \right)^2 \right) \, dx
\]
\[
= 2\pi \int_{0}^{2} \left( 5 - x^2 \right)^2 - \left( \frac{x^2}{4} \right)^2 \, dx
\]
\[
= 2\pi \int_{0}^{2} \left( 25 - 10x^2 + x^4 - \frac{x^4}{16} \right) \, dx
\]
\[
= 2\pi \int_{0}^{2} \left( 25 - 10x^2 + \frac{15x^4}{16} \right) \, dx
\]
\[
= 2\pi \left[ 25x - \frac{10x^3}{3} + \frac{15x^5}{5\cdot 16} \right]_0
\]
\[
= 2\pi \left( 25(2) - \frac{10(2)^3}{3} + \frac{3(2)^5}{16} \right)
\]
\[
= 2\pi \left( 50 - \frac{80}{3} + 6 \right) = \frac{176\pi}{3}
\]

7. \(64\pi/15\)

8. \(y = x^{2/3}, \ x = 1, \ y = 0\); about the \(y\)-axis.

\[y = x^{2/3} \iff x = y^{3/2}\]
\[ V = \int_{0}^{1} \pi \left( y^{3/2} \right)^2 \, dy = \pi \int_{0}^{1} y^3 \, dy \]

\[ = \pi \left[ \frac{y^4}{4} \right]_0^1 = \frac{\pi}{4} \]