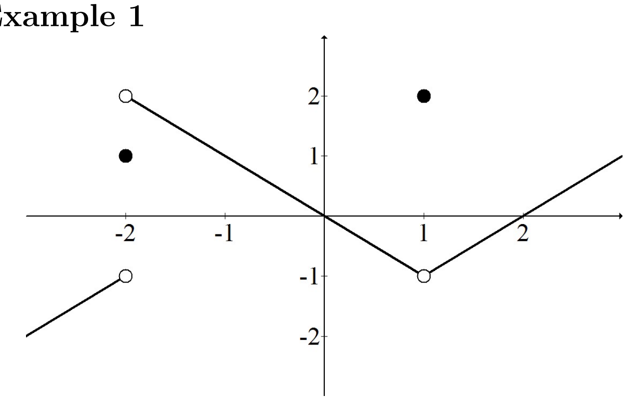
2 Limits and Derivatives

2.5 Continuity

We want to define what it means for a function to be continuous on an interval. An informal definition is to say that a function f is continuous on an interval I if we do not need to pick up our pen when we are sketching the graph. The pen continuous along the page.

Example 1



In this example, I have drawn a curve that is the graph of a function f. By the way, how do I know this curve is the graph of a function? Because it passes the vertical line test.

We see that the function is not continuous when $x = -2$ and $x = 1$. I say this because I had to pick up my pen at those places when I sketched the graph.

Now I will introduce a more formal definition for a function to be continuous at a number $x = a$. This is a definition that I require students to memorize.

Three Part Definition of Continuity A function f is continuous at a number a if

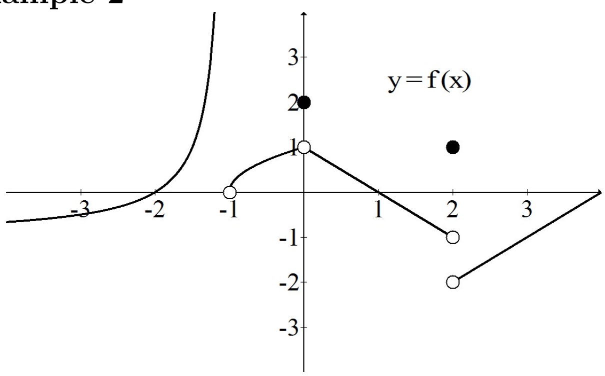
1. $f(a)$ is defined.

2. $\lim_{x \to a} f(x)$ exists.

3. $\lim_{x \to a} f(x) = f(a)$.

Please memorize this definition for the exam.

Example 2



At which values of x is the function not continuous. Which requirement(s) fail in the three part definition of continuity?

SOLUTION

1. At $x = -1$ because $\lim_{x \to -1} f(x)$ does not exist.

2. At $x = 0$ because $f(0) \neq \lim_{x \to 0} f(x)$.

   In this case, we say that f has a **removable discontinuity** at $x = 0$ because we can fix the continuity by redefining the function f at the number 0.

3. At $x = 2$ because $\lim_{x \to 2} f(x)$ does not exist. □

ENDPOINTS: We can adapt the definition of continuity to address endpoints of a function on a closed interval. When dealing with endpoints and limits at endpoints, we should only take a one sided limits. At endpoints it only makes sense to take either a left-hand limit or a right-hand limit, depending on whether the endpoint is a right or left endpoint. Except for this, the definition of continuity at an endpoint is the same as the definition of continuity at a point within the interior of an interval.

Instead of always saying that a function is not continuous at a number a, we often say that a function is **discontinuous** at a number a.

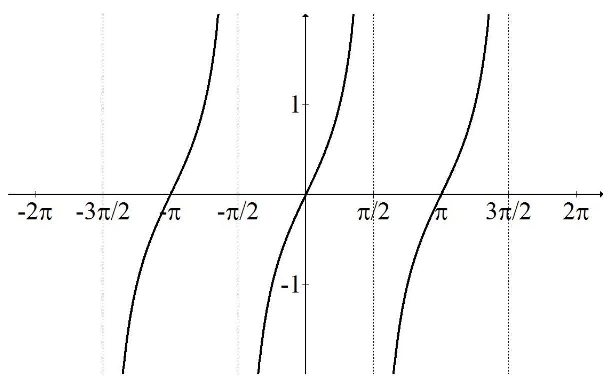
We say that a function is continuous on an interval I if it is continuous at every number a in the interval I.

We often just say that a function is continuous. This is meant to mean that it is continuous for all real numbers. We also sometimes say that a function is continuous on its domain. This means that as long as $f(a)$ exists, it is continuous at $x = a$.

An example is the function $y = \tan x$. The only places where we need to pick up the pen to draw the graph is where the function is undefined. On its domain, which is all the numbers where the function is defined, the function is continuous.

Suppose that we want to find the limit of a continuous function at a number a in its domain. This is easy. All we have to do is substitute the value a into the function. This is what it means for a function to be continuous.

$y = \tan x$



For example, we just said that $y = \tan x$ is continuous on its domain. So that if we want to find the limit of the function at say $x = \pi/4$, we just need to substitute that number $\pi/4$ in for x. That is,

$$\lim_{x \to \pi/4} \tan x = \tan(\pi/4) = 1.$$

We can also prove using the Limit Laws that combinations of continuous functions are continuous. So, for example, if we know that $y = \tan x$ and $y = \sin x$ are both continuous at $x = \pi/4$, then it will follow that $y = \tan x + \sin x$ is continuous at $x = \pi/4$. We state this in a theorem below.

Theorem 2.1 If f and g are continuous at a, and c is a constant, then the following functions are continuous at a:

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\frac{f}{g}$ provided $g(a) \neq 0$.

What about the composition of two continuous functions? In fact, the composition of two continuous functions will also be continuous. This fact is usually given as a separate theorem because it is both important and a little more difficult to prove. The proof is not given here.

Theorem 2.2 If f and g are continuous functions, then $y = f(g(x))$ is also continuous.

This is provided that we only consider values of x where $g(x)$ lies in the domain of f.

All of the elementary functions are continuous on there domain. This is stated in the next theorem.

Theorem 2.3 The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions
- inverse trigonometric functions

- exponential functions
- logarithmic functions
- absolute value function

Example 3 Find the limits of the following functions.

1. $\lim_{x \to \pi/6} 5 + \sin^2 x$
2. $\lim_{x \to \pi/4} \ln(\tan x)$

SOLUTION

1. $\lim_{x \to \pi/6} 5 + \sin^2 x = 5 + \sin^2 \pi/6 = 5 + (1/2)^2 = 5 + 1/4 = 21/4$
2. $\lim_{x \to \pi/4} \ln(\tan x) = \ln(\tan \pi/4) = \ln 1 = 0$ □

The Intermediate Value Theorem

Now we will give a theorem which is very important, but at the same time very elementary.

Here is the idea. Suppose that f is a continuous function. Moreover, suppose that $f(1)$ is a negative number and that $f(10)$ is a positive number. Then somewhere between $x = 1$ and $x = 10$, the function f must be equal to zero.

Here is another example. If it is 68° outside at 7 a.m. and 74° outside at 2 p.m., then at some time between 7 a.m. and 2 p.m., the temperature must have been 70°.

Theorem 2.4 The Intermediate Value Theorem Suppose that f is continuous on a closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

Corollary If f is continuous on the interval $[a, b]$ and $f(a)$ and $f(b)$ are both nonzero and have opposite signs (one is positive, the other negative), then there is a number c in the interval (a, b) such that $f(c) = 0$.

The following is a classic example that uses the Intermediate Value Theorem.

Example 4 Show that there is at least one real root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0.$$

SOLUTION For a large enough positive value of x, the function f will be positive. Let's evaluate $f(2)$.

$$f(2) = 4(2)^3 - 6(2)^2 + 3(2) - 2 = 32 - 24 + 6 - 2 = 12$$

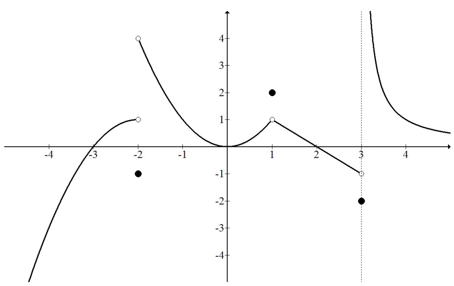
On the other hand, for a large enough negative value of x, the function f will be negative. In fact $f(0) = -2$.

Therefore, by the Intermediate Value Theorem, There is some number c between 0 and 2 such that $f(c) = 0$.

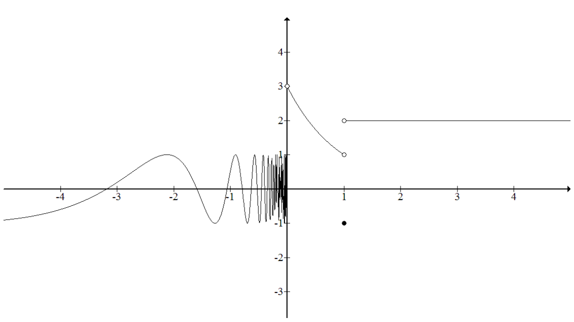
Homework

1. State the three part definition of continuity.

2. Give the values where the function is discontinuous. State which requirements from the three part definition of continuity fail.



3. Give the values where the function is discontinuous. State which requirements from the three part definition of continuity fail.



4. Sketch the graph of the function. Give the values where the function is discontinuous. State which requirements from the three part definition of continuity fail. $$f(x) = \frac{1}{x-1} \text{ at } x = 1$$

5. Sketch the graph of the function. Give the values where the function is discontinuous. State which requirements from the three part definition of continuity fail.
$$f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3 & \text{if } x = 2 \\ 3x-2 & \text{if } x > 2 \end{cases}$$

6. Determine the values of a and b that make the given function continuous.
$$f(x) = \begin{cases} \frac{2\sin x}{x} & \text{if } x < 0 \\ a & \text{if } x = 0 \\ b\cos x & \text{if } x > 0 \end{cases}$$

Use continuity to evaluate the limit of the function.

7. $\displaystyle\lim_{x\to 4} \frac{5+\sqrt{x}}{\sqrt{5+x}}$

8. $\displaystyle\lim_{x\to \pi} \sin(x + \sin x)$

The Intermediate Value Theorem

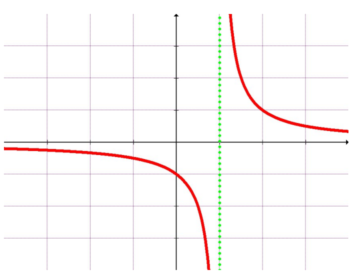
9. Let $f(x) = x^5 + 7x^4 + 5x^3 + 2x^2 + 9x - 10$. A table of values of this function is given below.

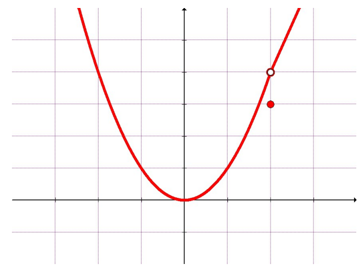
x	$f(x)$
-8	-6610
-7	-1690
-6	224
-5	620
-4	434
-3	170
-2	20
-1	-16
0	-10
1	14
2	200
3	980

Circle the interval(s) which must contain a zero of the function f.

$[-8,-7], [-7,-6], [-6,-5], [-5,-4], [-4,-3],$
$[-3,-2], [-2,-1], [-1,0], [0,1,], [1,2], [2,3]$

Homework Solutions

1. A function f is continuous at a number a if
   (a) $f(a)$ is defined
   (b) $\lim_{x\to a} f(x)$ exists
   (c) $\lim_{x\to a} f(x) = f(a)$
2. (a) Discontinuous at $x = -2$ because $\lim_{x\to -2} f(x)$ DNE
   (b) Discontinuous at $x = 1$ because $\lim_{x\to 1} f(x) \neq f(1)$
   (c) Discontinuous at $x = 3$ because $\lim_{x\to 3} f(x)$ DNE
3. (a) Discontinuous at $x = 0$ because $\lim_{x\to 0} f(x)$ DNE
   (b) Discontinuous at $x = 1$ because $\lim_{x\to 1} f(x)$ DNE
4. 

   Discontinuous at $x = 1$ because $\lim_{x\to 1} f(x)$ DNE
5. 

   Discontinuous at $x = 2$ because $\lim_{x\to 2} f(x) =$ 4 and $f(2) = 3$, therefore $\lim_{x\to 2} f(x) \neq f(2)$
6. $a = 2,\ b = 2$
7. $7/3$
8. 0
9. $[-7, 6], [-2, 1], [0, 1]$