2 Limits and Derivatives

2.3 Calculating Limits Using the Limit Laws

In this section we will first state some of the properties of limits that will allow us to find the limits of combination of functions. It is not necessary to memorize these properties. Instead, you should have a working knowledge of the properties so that you can evaluate limits.

Limit Laws Suppose that c is a number and $f(x)$ are two functions whose limits exist at the number a. That is $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist. Then the following is properties are true.

1. $\lim_{x\to a} c = c$

   The idea is that a constant always remains the same regardless of the value of x, so that as x approaches a, the constant value is always the same.

   Example 1 $\lim_{x\to 5} 2 = 2$

2. $\lim_{x\to a} x = a$

   We can say "As x approaches a, the function $y = x$ approaches a."

   Example 2 $\lim_{x\to 5} x = 5$

3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$

   If we already know the limit of a function, what will happen to the limit when we multiply the function by a number? The answer is that we just multiply the limit by that same number.

   Example 3 $\lim_{x\to 5} 7x = 7\lim_{x\to 5} x = 7 \cdot 5 = 35$

4. $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

   If we know the limits of two functions f and g, then the limit of $f + g$ is just found by adding the two limits.

   Example 4 $\lim_{x\to 5}(7x+2) = \lim_{x\to 5} 7x + \lim_{x\to 5} 2 = 7\lim_{x\to 5} x + \lim_{x\to 5} 2 = 7\cdot 5 + 2 = 37$

5. $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

   Example 5 $\lim_{x\to 5}(7x-2) = \lim_{x\to 5} 7x - \lim_{x\to 5} 2 = 7\lim_{x\to 5} x - \lim_{x\to 5} 2 = 7\cdot 5 - 2 = 33$

6. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$

   If we know the limit of f and the limit of g as x approaches a, then the limit of f times g is found by multiplying the limits of the two functions.

   Example 6 $\lim_{x\to 5}(7x-2)(7x+2) = \lim_{x\to 5}(7x-2)\lim_{x\to 5}(7x+2) = 37 \cdot 33 = 1221$

7. $[\lim_{x\to a}[f(x)]^n = [\lim_{x\to a}]^n$

   We can now find the limit of $y = x^n$ as x approaches a number a.

   Example 7 $\lim_{x\to 5} x^3 = \left(\lim_{x\to 5} x\right)^3 = 5^3 = 125$

   One of the results of these properties is that we can easily find the limit of polynomials.

   Example 8

$$\begin{aligned}\lim_{x\to 5}(4x^2 + 7x + 6) &= \lim_{x\to 5} 4x^2 + \lim_{x\to 5} 7x + \lim_{x\to 5} 6 \\ &= 4\lim_{x\to 5} x^2 + 7\lim_{x\to 5} x + \lim_{x\to 5} 6 \\ &= 4\left(\lim_{x\to 5} x\right)^2 + 7\lim_{x\to 5} x + \lim_{x\to 5} 6 \\ &= 4\cdot(5)^2 + 7\cdot(5) + 6 = 141\end{aligned}$$

   We see that to find the limit of a polynomial at a number a, all we have to do is directly substitute a into the polynomial.

   Example 9 $\lim_{x\to 5}(x^3 + 7x + 1) = (5)^3 + 7(5) + 1 = 125 + 35 + 1 = 161$

   Let's move on to the next property.

8. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0$.

We can now easily find the limit of rational functions, which are defined as the fraction of two polynomials.

Example 10 $\lim_{x\to 5} \frac{x^2+7x+1}{x-3} = \frac{\lim_{x\to 5}(x^2+7x+1)}{\lim_{x\to 5}(x-3)}$

Because the numerator and denominator of a rational function are polynomials, we can use direct substitution to calculate the limits.

$= \frac{(5)^2+7(5)+1}{(5)-3} = \frac{61}{2}$

We next have two properties that allow us to find the limit of root functions using direct substitution.

9. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$

10. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$

Example 11 $\lim_{x\to 5} \sqrt{x^2+7x+4} = \sqrt{(5)^2+7(5)+4} = \sqrt{64} = 8$

We can state the following theorem regarding the limits of polynomials and rational functions.

Theorem 2.1 Direct Substitute Property If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a)$$

It is important that the value of a be in the domain of the rational function. Direct substitution will not work for rational functions if the result is a fraction where zero is in the denominator. A fraction with zero in the denominator is undefined.

Next we move on to more challenging problems where direct substitution will not work at the beginning because this would result in a zero in the denominator of a fraction.

Example 12 A Limit Where Two Factors Cancel Evaluate the limit

$$\lim_{x \to -3} \frac{3x+9}{x^2-9}$$

SOLUTION

The problem is that even though this is a rational function, the number -3 is not in the functions domain. Direct substitution would give a zero in the denominator.

However, substituting -3 into the function does give us some information: We see that -3 is a zero of both the numerator and the denominator. This means that both the numerator and the denominator must have $(x+3)$ as a factor, and that this factor will cancel.

$$\lim_{x \to -3} \frac{3x+9}{x^2-9} = \lim_{x \to -3} \frac{3(x+3)}{(x+3)(x-3)}$$

$$= \lim_{x \to -3} \frac{3}{x-3}$$

The last limit can be found by directly substituting the number $x = -3$. This works because the function $y = \dfrac{3}{x-3}$ is, unlike the original function, defined at $x = -3$. Finally the answer is

$$\lim_{x \to -3} \frac{3x+9}{x^2-9} = \lim_{x \to -3} \frac{3}{x-3} = \frac{3}{-3-3} = -\frac{1}{2}$$

In the next example, direct substitution will give a form of $\frac{0}{0}$ which indicates that some simplification might be possible to find the limit. This is a special example where a technique is used for functions involving radicals.

Example 13 Finding a Limit by Rationalizing Evaluate

$$\lim_{x \to 0} \frac{\sqrt{x^2+9}-3}{x^2}$$

SOLUTION

$$\lim_{x \to 0} \frac{\sqrt{x^2+9}-3}{x^2}$$

$$
\begin{aligned}
&= \lim_{x\to 0} \frac{\sqrt{x^2+9}-3}{x^2}\cdot\frac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3}\\
&= \lim_{x\to 0} \frac{(\sqrt{x^2+9})^2-(3)^2}{x^2(\sqrt{x^2+9}+3)}\\
&= \lim_{x\to 0} \frac{(x^2+9-9}{x^2(\sqrt{x^2+9}+3)}\\
&= \lim_{x\to 0} \frac{(x^2}{x^2(\sqrt{x^2+9}+3)}\\
&= \lim_{x\to 0} \frac{1}{\sqrt{x^2+9}+3}\\
&= \frac{1}{\sqrt{0^2+9}+3}\\
&= \frac{1}{(\sqrt{9}+3)} = \frac{1}{6}
\end{aligned}
$$

We will say that a function f has the direct substitution property at a number a if

$$\lim_{x\to a} f(x) = f(a)$$

Direct Substitution can be used to evaluate most of the elementary functions, as long as the limit is taken at a number <u>where the function is defined</u>.

In the previous section, we found the limit by sketching the graph of a function. Unless there was a hole in the graph, the limit of a function at a value a could be found by Direct Substitution. Now think of the graphs of the elementary functions. None of these functions have any jumps or holes in the graph. Some of these functions have vertical asymptotes at undefined numbers, but these undefined numbers are not in the domain of the functions.

Therefore, we have the following theorem which says that elementary functions have the Direct Substitution Property at a value a as long as the value is in the domain of the function. No proof is given, except that we can refer to the graph of the function.

Theorem 2.2 Direct Substitution The following functions have the Direct Substitution Property for all numbers a <u>in their domain</u>.

1. Polynomials
2. Rational Functions
3. Root Functions
4. Trigonometric Functions
5. Exponential Functions
6. Logarithmic Functions
7. Inverse Trigonometric Functions
8. Absolute Value Function

Moreover, the composition of these functions also have the Direct Substitution Property at a number a provided that the number a is in the domain of the composition function.

Another name for the Direct Substitution Property is to say that a function is continuous. This will be the subject of the next section.

Example 14 Evaluate the limit of the following function.

$$\lim_{x\to 0} \tan^{-1}(e^x)$$

SOLUTION We can find the limit by direct substitution.

$$\lim_{x\to 0} \tan^{-1}(e^x) = \tan^{-1}(e^0) = \tan^{-1} 1 = \pi/4.$$

In the next example, we look at a limit involving absolute value. The trick is to write the function as a piecewise defined function without the absolute value symbol.

Example 15 An Example Involving Absolute Value

Find the limits, if they exist.

1. $\lim_{x\to 0^+} \dfrac{x}{|x|}$

2. $\lim_{x \to 0^-} \frac{x}{|x|}$

3. $\lim_{x \to 0} \frac{x}{|x|}$

SOLUTION Note:

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

1. $\lim_{x \to 0^+} \frac{x}{|x|} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1$

2. $\lim_{x \to 0^-} \frac{x}{|x|} = \lim_{x \to 0^+} \frac{x}{-x} = \lim_{x \to 0^-} -1 = -1$

3. $\lim_{x \to 0} \frac{x}{|x|}$ does not exist because the left hand and right hand limits do not agree.

We present a theorem that gives another method for finding the limit of a function. The idea is that we might have a function that is not an elementary function and which we cannot simplify or use the Direct Substitution Property. However, if we can compare it to a function that is easier to work with, we can find the limit of the function.

Theorem 2.3 The Squeeze Theorem
If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L.$$

then

$$\lim_{x \to a} g(x) = L.$$

Example 16 Evaluate the following limit.

$$\lim_{x \to 0} x^2 \sin \frac{1}{x}$$

Solution First we find an inequality for $\sin\frac{1}{x}$.

$$-1 \leq \sin\frac{1}{x} \leq 1$$

Next multiply through by x^2. This expression is never negative, so we do not have to worry about changing the direction of the inequality.

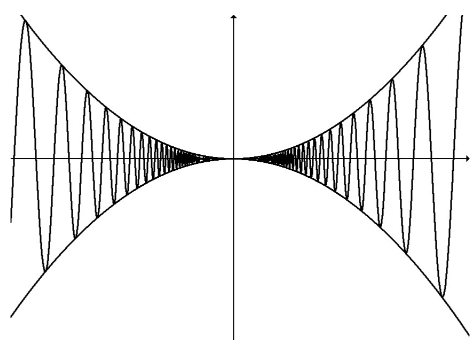
$$-x^2 \leq x^2 \sin\frac{1}{x} \leq x^2$$

We can find the limit of the two outer functions as x approaches 0.

$$\lim_{x\to 0} -x^2 = 0, \quad \lim_{x\to 0} x^2 = 0$$

Therefore, by the Squeeze Theorem, $\displaystyle\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$.

Below is a graph of the function $y = x^2\sin(1/x)$. Also shown on the same plane are the functions $y = x^2$ and $y = -x^2$. One can see that the function $y = x^2\sin(1/x)$ is bounded by the two functions $y = x^2$ and $y = -x^2$.



Homework. Evaluate the indicated limit, if it exists.

1. $\lim_{x\to -7}(2x+5)$
2. $\lim_{x\to 12}(10-3x)$
3. $\lim_{x\to 2}(-x^2+5x-2)$
4. $\lim_{x\to 2}\frac{x+3}{x+6}$
5. $\lim_{x\to 5}\frac{4}{x-3}$
6. $\lim_{x\to -5}\frac{x^2}{5-x}$
7. $\lim_{x\to 2}\frac{x^2-x-2}{x^2-4}$
8. $\lim_{x\to -2}\frac{x+2}{x^2+5x+6}$
9. $\lim_{x\to 3}\frac{x^2-9}{x^2-5x+6}$
10. $\lim_{x\to 1}\frac{x-1}{x^3-1}$
11. $\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}$
12. $\lim_{x\to 0}\frac{2x}{3-\sqrt{x+9}}$
13. $\lim_{x\to 1}\frac{x-1}{\sqrt{x}-1}$
14. $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$
15. $\lim_{x\to 2}\frac{\sqrt{x^2+12}-4}{x-2}$
16. $\lim_{x\to -2}\frac{x+2}{\sqrt{x^2+5}-3}$
17. $\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{2}{x^2-1}\right)$
18. $\lim_{x\to 2}\left(\frac{1}{x-2}-\frac{3}{x^2-x-2}\right)$
19. $\lim_{t\to -2}\frac{\frac{1}{2}+\frac{1}{t}}{2+t}$
20. $\lim_{x\to -4}\frac{\frac{1}{4}+\frac{1}{x}}{4+x}$
21. Use $\lim_{x\to a}f(x)=2$, $\lim_{x\to a}g(x)=-3$ and $\lim_{x\to a}h(x)=0$ to determine the limit, if possible.
    (a) $\lim_{x\to a}[2f(x)-3g(x)]$
    (b) $\lim_{x\to a}\left[\frac{f(x)+g(x)}{h(x)}\right]$
22. Find the limit, if it exists. If the limit does not exist, explain why.
    Note:
    $$|x|=\begin{cases} x & \text{if} \quad x\geq 0 \\ -x & \text{if} \quad x<0 \end{cases}$$
    (a) $\lim_{x\to 0^-}\left(\frac{1}{x}-\frac{1}{|x|}\right)$
    (b) $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{|x|}\right)$
23. The Squeeze Theorem: If $4x-9\leq f(x)\leq x^2-4x+7$ for $x\geq 0$, find $\lim_{x\to 4}f(x)$.
24. The Squeeze Theorem: If $2x\leq g(x)\leq x^4-x^2+2$ for all x, find $\lim_{x\to 1}g(x)$.

Homework Solutions

1. -9
2. -26
3. 4
4. 5/8
5. 2
6. 5/2
7. 3/4
8. 1
9. 6
10. 1/3
11. 1/4
12. -12
13. 2
14. 1/4
15. 1/2
16. $-3/2$
17. 1/2
18. 1/3
19. $-1/4$
20. $-1/16$
21. a) 13 b) DNE
22. a) DNE b) 0
23. 7
24. 2