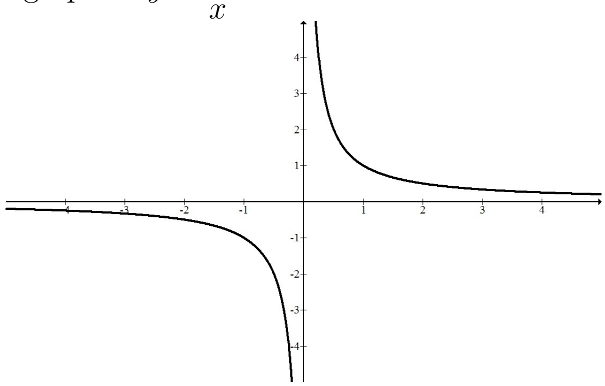
2 Limits and Continuity

2.6 Limits at Infinity; Horizontal Asymptotes

The graph of $y = \frac{1}{x}$ is shown below.

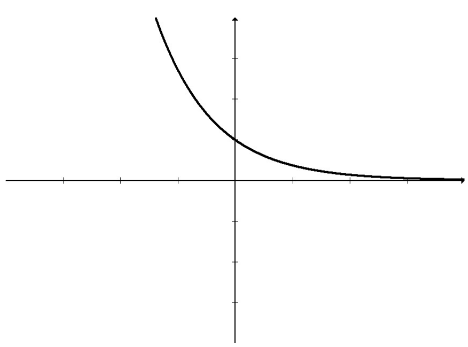


We can look at the behavior of the function as x becomes large. As x becomes large, we see that the function approaches zero. Notice that the function is never equal to zero, because the numerator is never 0. We write

$$\lim_{x \to \infty} \frac{1}{x} = 0 \text{ and } \lim_{x \to -\infty} \frac{1}{x} = 0$$

Example 1 Sketch the graph of $f(x) = e^{-x}$. Use the graph to evaluate $\lim_{x \to \infty} e^{-x}$.

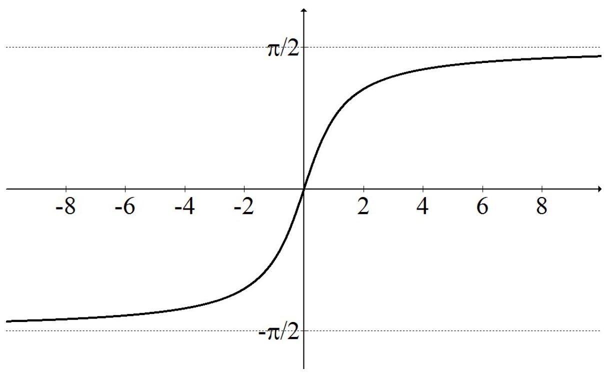
SOLUTION



By looking at the graph, we see that $\lim_{x \to \infty} e^{-x} = 0$. $\square$

Example 2 Sketch the graph of $f(x) = \tan^{-1} x$. Use the graph to evaluate $\lim_{x \to \infty} \tan^{-1} x$.

SOLUTION



By looking at the graph, we see that $\lim_{x \to \infty} \tan^{-1} x = \pi/2$. □

Here we give a formal definition of limit as x goes to infinity.

Definition Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the value of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

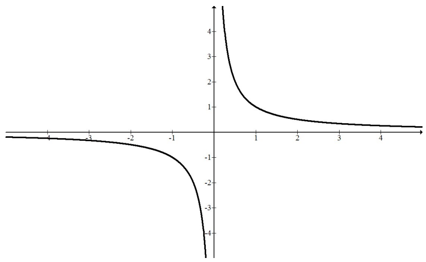
In a similar way we can define the limit of a function f as x goes to negative infinity.

Definition Let f be a function defined on some interval $(-\infty, a)$. Then

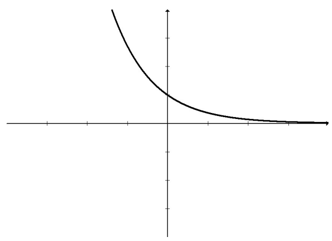
$$\lim_{x \to -\infty} f(x) = L$$

means that the value of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.

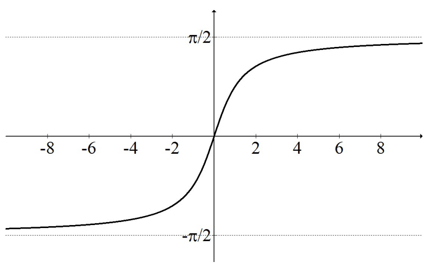
If $\lim_{x \to \infty} f(x) = L$, then the line $y = L$ is a **horizontal asymptote**. The functions in the previous examples all had horizontal asymptotes.



$$\lim_{x\to\infty} \frac{1}{x} = 0, \text{ Horizontal Asymptote } y = 0$$



$$\lim_{x\to\infty} e^{-x} = 0, \text{ Horizontal Asymptote } y = 0$$



$$\lim_{x\to\infty} \tan^{-1} x = \pi/2, \text{ Horizontal Asymptote } y = \pi/2$$

We are often finding the limits as x approaches infinity of functions of the form $y = \dfrac{1}{x^r}$. We saw that the function $y = \dfrac{1}{x} \to 0$ as $x \to \infty$. The same is true for the functions $y = \dfrac{1}{x^2}$ and the function $y = \dfrac{1}{x^3}$. This leads to the following theorem.

Theorem 2.1 If $r > 0$ is a rational number, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0.$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0.$$

Example 3 Find the following limits.

1. $\lim_{x\to\infty} \frac{1}{x^2}$

2. $\lim_{x\to-\infty} \frac{5}{x^3}$

3. $\lim_{x\to\infty} \frac{5}{\sqrt{x}}$

SOLUTION All of these have limit of 0.

The Limit Laws apply to limits where x goes to infinity or negative infinity. We can use these Limit Laws to evaluate the limits of rational functions as x goes to infinity or negative infinity.

Example 4 Find the limit.

$$\lim_{x\to\infty} \frac{3x^4 + 5x^2 + 1}{x^4 + 6}$$

SOLUTION There is a trick (should I say "technique") to solving this problem. We divide the numerator and the denominator by the highest power in the denominator.

$$\begin{aligned}
\lim_{x\to\infty} \frac{3x^4 + 5x^2 + 1}{x^4 + 6} &= \lim_{x\to\infty} \left(\frac{3x^4 + 5x^2 + 1}{x^4 + 6}\right) \cdot \left(\frac{1/x^4}{1/x^4}\right) \\
&= \lim_{x\to\infty} \frac{3x^4/x^4 + 5x^2/x^4 + 1/x^4}{x^4/x^4 + 6/x^4} \\
&= \lim_{x\to\infty} \frac{3 + 5/x^2 + 1/x^4}{1 + 6/x^4} \\
&= \frac{3 + 0 + 0}{1 + 0} = 3
\end{aligned}$$

In the last step, the limit of each individual term inside the fraction was taken. □

The Behavior of Polynomials as $x \to \infty$

Next we will study the behavior of polynomials as x goes to infinity. We begin with the polynomial $f(x) = x^2$. As x approaches infinity, the function approaches infinity. As x approaches $-\infty$, the function will approach infinity. This is because a large number squared is a large positive number. We write

$$\lim_{x \to \infty} x^2 = \infty \text{ and } \lim_{x \to -\infty} x^2 = \infty$$

Example 5 Find the following infinite limits.

1. $\lim\limits_{x \to \infty} x^3$
2. $\lim\limits_{x \to -\infty} x^3$
3. $\lim\limits_{x \to -\infty} x^4$

SOLUTION 1. ∞, 2. $-\infty$, 3. ∞.

What about a general polynomial? A general rule of thumb is that as x becomes large, it behaves like its highest power term. For example, consider the function $f(x) = x^5 + x^4 + x^3 + 1$. For x not equal to zero, we can factor out the highest power term.

$$f(x) = x^5 + x^4 + x^3 + 1 = x^5(1 + 1/x + 1/x^2 + 1/x^4)$$

The term $1/x$ has limit 0 as $x \to \infty$, as do the other terms $1/x^2$ and $1/x^4$. We see then that

$$\lim_{x \to \infty} x^5 + x^4 + x^3 + 1 = \lim_{x \to \infty} x^5 = \infty$$

Example 6 Find the limit.

$$\lim_{x \to -\infty} \left(-7x^5 + 6x^3 + x^2 + 1\right)$$

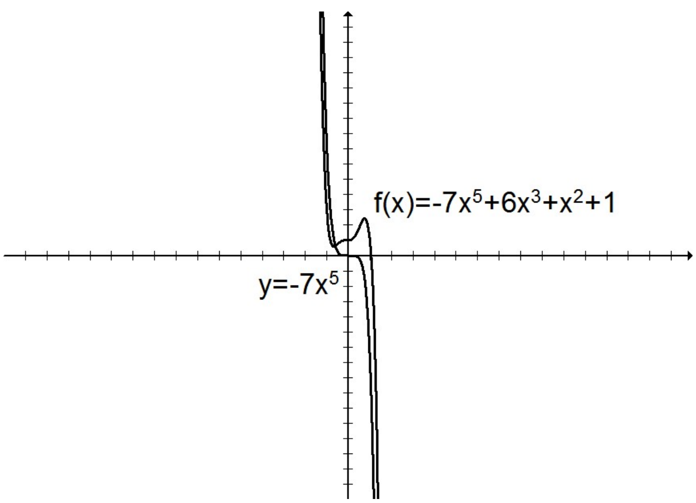
SOLUTION

$$\lim_{x \to -\infty} \left(-7x^5 + 6x^3 + x^2 + 1\right) = \lim_{x \to -\infty} -7x^5 = \infty$$

Note that while x^5 becomes a large negative number, it is multiplied by -7 which makes the final answer a large positive number. □

In fact, as x becomes very large, the graph of $f(x) = (-7x^5 + 6x^3 + x^2 + 1)$ looks very much like the graph of $y = -7x^5$. If we take the quotient of the two functions, we see that the limit as x goes to infinity is 1, which indicates that the two functions are very close to each other for large x.

$$\lim_{x\to\infty} \frac{-7x^5 + 6x^3 + x^2 + 1}{-7x^5} = \lim_{x\to\infty} \left(1 - \frac{6}{7x^2} - \frac{1}{7x^3} - \frac{1}{7x^5}\right) = 1$$



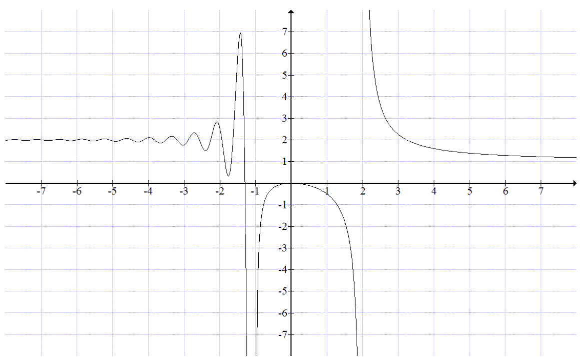
The graphs of
$f(x) = -7x^5 + 6x^3 + x^2 + 1$
and $y = -7x^5$ are very close to each other when x is a large number.

Homework

1. For the function graphed below, state the following.

   (a) $\lim_{x \to -\infty} f(x)$

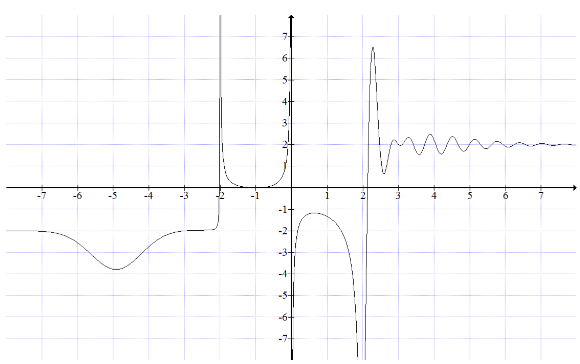
   (b) $\lim_{x \to \infty} f(x)$



2. For the function graphed below, state the following.

   (a) $\lim_{x \to -\infty} f(x)$

   (b) $\lim_{x \to \infty} f(x)$



3. Sketch the graph of an example of a function f that satisfies all of the given conditions.

$$f(0) = 0,\ f(1) = 1,\ \lim_{x\to\infty} f(x) = 0,\ f \text{ is odd}$$

4. Sketch the graph of an example of a function f that satisfies all of the given conditions.

$$\lim_{x\to 0^+} f(x) = \infty,\ \lim_{x\to 0^-} f(x) = -\infty,$$

$$\lim_{x\to\infty} f(x) = 1,\ \lim_{x\to-\infty} f(x) = 1$$

5. Sketch the graph of an example of a function f that satisfies all of the given conditions.

$$\lim_{x\to 2} f(x) = -\infty,\ \lim_{x\to\infty} f(x) = \infty,$$

$$\lim_{x\to-\infty} f(x) = 0,$$

$$\lim_{x\to 0^+} f(x) = \infty,\ \lim_{x\to 0^-} f(x) = -\infty$$

6. Sketch the graph of an example of a function f that satisfies all of the given conditions.

$$\lim_{x\to -2} f(x) = \infty,\ \lim_{x\to-\infty} f(x) = 3,$$

$$\lim_{x\to\infty} f(x) = -3$$

Determine each limit. Your answer should either be a number, ∞, $-\infty$, or does not exist.

7. $\displaystyle\lim_{x\to\infty} \frac{x^2+3}{5x^2+3x+2}$

8. $\displaystyle\lim_{x\to-\infty} \frac{(x-6)(x+2)}{(x+1)(x+7)(x-4)}$

9. $\displaystyle\lim_{x\to-\infty} \frac{(x-6)(x+2)(x+7)}{(x+1)(x-4)}$

10. $\displaystyle\lim_{x\to\infty} \frac{x}{\sqrt{4+x^2}}$

11. $\displaystyle\lim_{x\to\infty} \frac{x+2}{\sqrt{9x^2+1}}$

12. $\displaystyle\lim_{x\to-\infty} 7x^5+6x^4+3x^3+2x+2$

13. $\displaystyle\lim_{x\to\infty} x^5-x^4$

14. $\displaystyle\lim_{x\to\infty} (x-\sqrt{x})$

15. $\displaystyle\lim_{x\to\infty} e^{-x}$

16. $\displaystyle\lim_{x\to\infty} \tan^{-1} x$

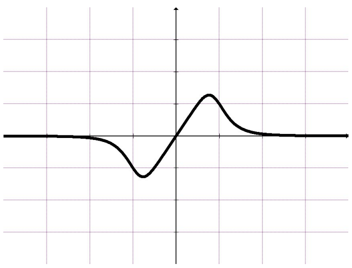
17. $\displaystyle\lim_{x\to\infty} \frac{\sin^2 x}{x^2}$

Homework Solutions

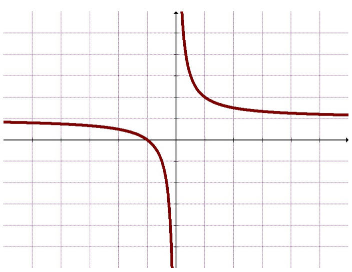
1. (a) 2 (b) 1

2. (a) -2 (b) 2

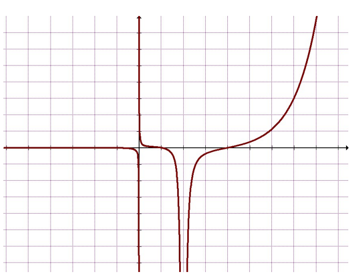
3.



4.



5.



6.

7. $1/5$

8. 0

9. $-\infty$

10. 1

11. $1/3$

12. $-\infty$

13. ∞

14. ∞

15. 0

16. $\pi/2$

17. 0