

## 2 Limits and Derivatives

### 2.2 The Concept of Limits and One-Sided Limits

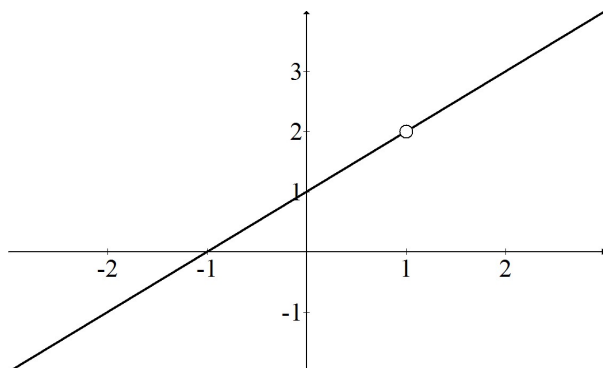
Let's begin by studying the function  $f(x) = \frac{x^2 - 1}{x - 1}$ .

This function is undefined when  $x = 1$  because if  $x = 1$  is substituted into the function, the result is a fraction with denominator 0, which is undefined.

Even though  $f(1)$  is not defined, we might want to know the value of  $f(x)$  when  $x$  is a number very close to 1.

One way to achieve this is to sketch the graph of the function. We see that

$$f(x) = \frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1} = x + 1, \quad \text{for } x \neq 1$$



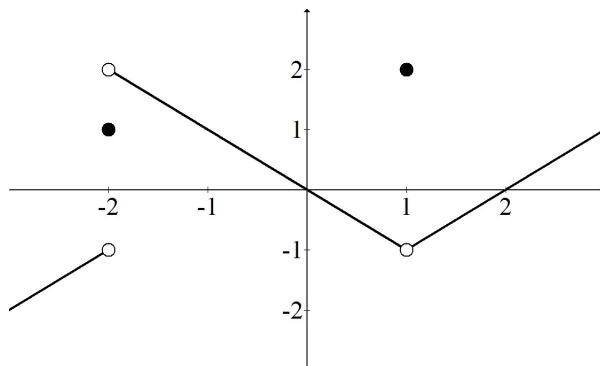
For all values of  $x$  not equal to 1, the function is equal the line  $y = x + 1$ . At  $x = 1$  the function is undefined. This is seen as a hole in the graph at the point  $(1, 2)$ .

By looking at the graph, we can answer the following question. As  $x$  approaches 1, what value does  $f(x)$  approach? Remember that  $y = f(x)$ , so we are asking what number the  $y$ -coordinate approaches as the  $x$ -coordinate approaches 1. The answer is 2.

We say that the limit of  $f(x)$  as  $x$  approaches 1 is the number 2. This is written as

$$\lim_{x \rightarrow 1} f(x) = 2$$

**Example 1** The graph of a function  $f$  is given below.



1. As  $x$  approaches the number 1, what number does  $y$  approach? The answer is  $-1$ .

We write this as

$$\lim_{x \rightarrow 1} f(x) = -1$$

2. Note that the  $y$ -value of the function at 1 is 2. That is,

$$f(1) = 2$$

3. As  $x$  approaches the number  $-2$  from the left, what number does  $y$  approach? The answer is  $-1$ .

We write this as

$$\lim_{x \rightarrow -2^-} f(x) = -1$$

The  $-$  symbol on the shoulder of  $-2$  means that  $x$  approaches  $-2$  from the left.

4. As  $x$  approaches the number  $-2$  from the right, what number does  $y$  approach? The answer is 2.

We write this as

$$\lim_{x \rightarrow -2^+} f(x) = 2$$

The + symbol on the shoulder of  $-2$  means that  $x$  approaches  $-2$  from the right.

5. As  $x$  approaches  $-2$ , what number does  $y$  approach? There is no answer. We say

$$\lim_{x \rightarrow -2} f(x) \text{ does not exist.}$$

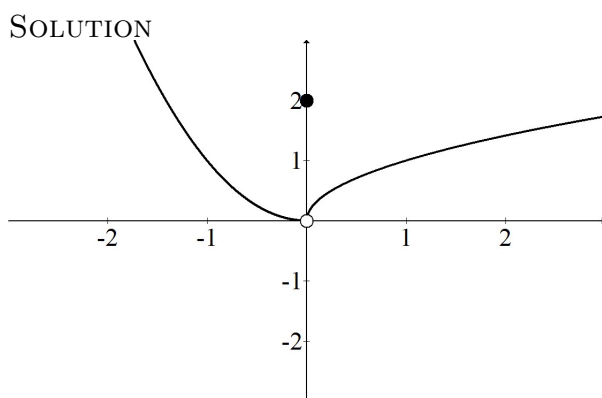
In order for the limit to exist, the left hand limit and the right hand limit must equal each other.

6. What is the  $y$ -value when  $x = -2$ ? The answer is 1. That is,

$$f(-2) = 1$$

**Example 2** By sketching the graph of the function  $f(x)$  given below, guess the value of  $\lim_{x \rightarrow 0} f(x)$ .

$$f(x) = \begin{cases} x^2 & \text{when } x < 0 \\ 2 & \text{when } x = 0 \\ \sqrt{x} & \text{when } x > 0 \end{cases}$$



$$\lim_{x \rightarrow 0} f(x) = 0 \quad \square$$

Here a definition of limit is given.

**Definition of Limit** We write

$$\lim_{x \rightarrow a} f(x) = L$$

and say “the limit of  $f(x)$ , as  $x$  approaches  $a$ , equals  $L$ ” if we can make the values of  $f(x)$  arbitrarily close to  $L$  (as close to  $L$  as we like) by taking  $x$  to be sufficiently close to  $a$  (on either side of  $a$ ) but not equal to  $a$ .

**Example 3 Approximating the Value of a Limit** Suppose we want to find the following limit.

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 9} - 3}{x^2}$$

We can estimate the value of a limit by making a table and plotting points. We can make the  $x$ -values closer and closer to 0 and try to see if the  $y$ -values seem to be approaching a number.

$x$	$\frac{\sqrt{x^2+9}-3}{x^2}$
$\pm 1.0$	0.16228
$\pm 0.5$	0.16553
$\pm 0.1$	0.16662
$\pm 0.05$	0.16666
$\pm 0.01$	0.16667

The table seems to show that the limit is the number  $0.1\bar{6} = 1/6$ . That is

$$\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 9} - 3}{x^2} = \frac{1}{6}$$

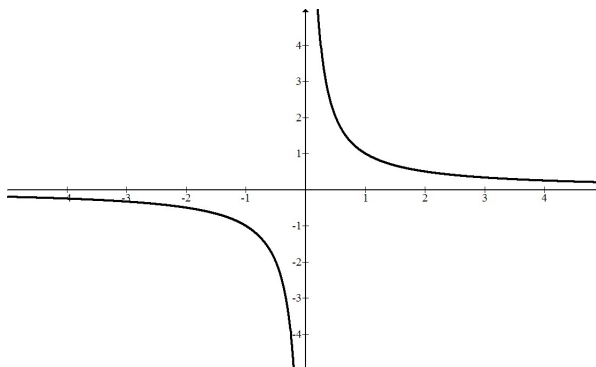
In this section we introduced the concept of limit. We saw that the limit can be found by sketching the graph of the function and by plotting points.

Plotting points to approximate the limit can sometimes give a wrong answer, and the graph of a function is not always accurate. For this reason, we will explore other techniques to find the limit in the next section.

**Limits that go to infinity.**

We begin by sketching the graph of the function  $y = \frac{1}{x}$ . To help us, let's plot some points.

$x$	$1/x$
1	1
2	1/2
3	1/3
4	1/4
10	1/10
1/2	2
1/3	3
1/10	10
-1	-1
-2	-1/2
-10	-1/10
-1/2	-2
-1/10	-10



The function  $y = 1/x$  is undefined at  $x = 0$ . It is at  $x = 0$  that the function has a vertical asymptote. As  $x$  approaches 0 from the right, we see that the function gets bigger and bigger. We write this as

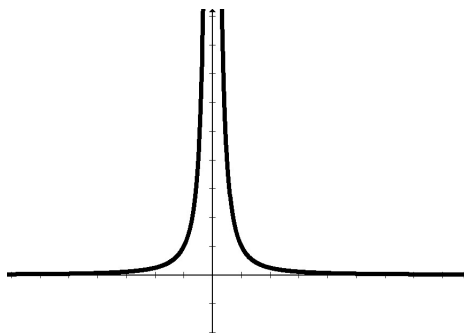
$$\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty$$

As  $x$  approaches 0 from the left, we see that the function gets bigger and bigger in magnitude, but the function is negative. We write this as

$$\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$$

**Example 4** Let  $f(x) = \frac{1}{x^2}$ .

1. Sketch the graph. Again we can plot some points. Keep in mind that the function is undefined at  $x = 0$ . The line  $x = 0$  is a vertical asymptote.



2. Find  $\lim_{x \rightarrow 0} \frac{1}{x^2}$ .

**SOLUTION** If direct substitution gives a form of  $\frac{1}{0}$ , then we should expect that the limit will either be  $\infty$  or  $-\infty$ . For limits of this form, the best way to proceed is by plotting points. We can plot numbers close to 0 from the left and from the right.

$x$	$1/x^2$
1	1
1/10	100
1/100	10,000
1/1000	1,000,000
-1	1
-1/10	100
-1/100	10,000
-1/1000	1,000,000

The answer is infinity.  $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$

Technically, if a function grows without bound, we say that the limit does not exist. However, if we see that the function becomes a larger and larger positive number, we say that the limit is  $\infty$ . If it is becoming a larger and larger negative number, we say that the limit is  $-\infty$ . The directions on the homework will ask the reader to find the infinite limit.

You can recognize these types of limit by first trying direct substitution. If the result is an expression of the form  $\frac{a}{0}$ , then most likely, the result will be an infinite limit. The way to determine whether the limit is positive or negative infinity is to plot points. It is often the case that the limit from the left and from the right will both be infinity, but one will be negative infinity and the other positive infinity. For this reason, it is necessary to plot points from both the left and the right.

Finally, we give a definition of an infinite limit.

**Definition** Let  $f$  be the function defined on both sides of  $a$ , except possibly at  $a$  itself. Then

$$\lim_{x \rightarrow a} f(x) = \infty$$

means that the values of  $f(x)$  can be made arbitrarily large (as large as we please) by taking  $x$  sufficiently close to  $a$ , but not equal to  $a$ .

**Example 5** Find the infinite limit.

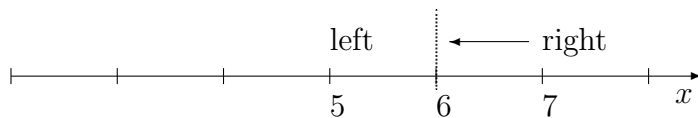
$$\lim_{x \rightarrow 6^+} \frac{x(x-1)}{x^2-5x-6}$$

**SOLUTION** This function would probably be difficult to graph. Instead, we can plot points. First, let's factor the denominator.

$$\lim_{x \rightarrow 6} \frac{x(x-1)}{x^2-5x-6} = \lim_{x \rightarrow \infty} \frac{x(x-1)}{(x-6)(x+1)}$$

We see that direct substitution gives a form of  $\frac{30}{0}$ . Now we can plot points. The question is whether the answer will be  $\infty$  or  $-\infty$ .

The biggest mistake people make is by plotting points from the wrong direction. We want numbers to the right of  $x = 6$ . It is often useful to sketch a number line to clarify which numbers are from the right.



$x$	$\frac{x(x-1)}{(x-6)(x+1)}$
7	$\frac{(7)(6)}{(1)(8)}$
6.1	$\frac{(6.1)(5.1)}{(.1)(7.1)} \approx \frac{30}{(.1)(7)}$
6.01	$\frac{(6.01)(5.01)}{(.01)(7.01)} \approx \frac{30}{(.01)(7)}$
6.001	$\frac{(6.001)(5.001)}{(.001)(7.001)} \approx \frac{30}{(.001)(7)}$

Keep in mind that dividing by a small positive number results in a large positive number. The final answer is

$$\lim_{x \rightarrow 6^+} \frac{x(x-1)}{x^2 - 5x - 6} = \infty$$

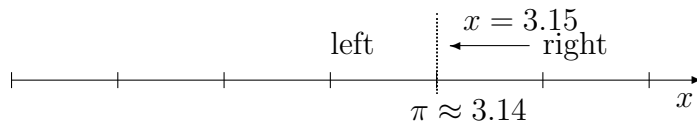
Next we show an example of a limit involving a trigonometric function.

**Example 6** Evaluate the infinite limit.

$$\lim_{x \rightarrow \pi^+} \csc x$$

**SOLUTION** First note that  $\csc x = \frac{1}{\sin x}$ . If we try direct substitution, we get  $\frac{1}{0}$ , which is undefined, but tells us that the answer is possibly either  $\infty$  or  $-\infty$ .

There are two different approaches that we can take. The first approach is to use a calculator to evaluate  $\csc x$  at values close to  $\pi$  from the right.



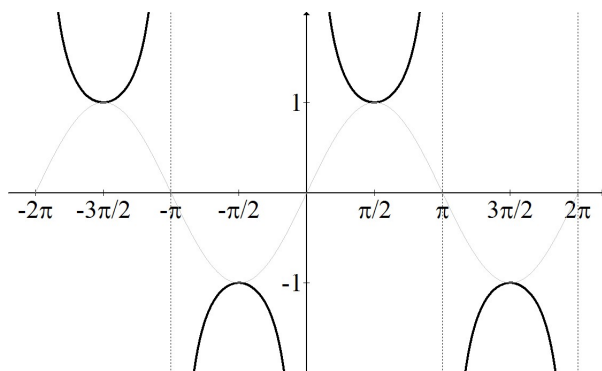
The number 3.15 is close to  $\pi$  and to the right of  $\pi$  on the number line. Make sure that your calculator is set to radians.

$$\csc(3.15) = \frac{1}{\sin(3.15)} \approx -118.9$$

Therefore, I conclude that

$$\lim_{x \rightarrow \pi^+} \csc x = -\infty$$

Another approach is to sketch the graph of  $y = \csc x$  and determine the limit from the graph.

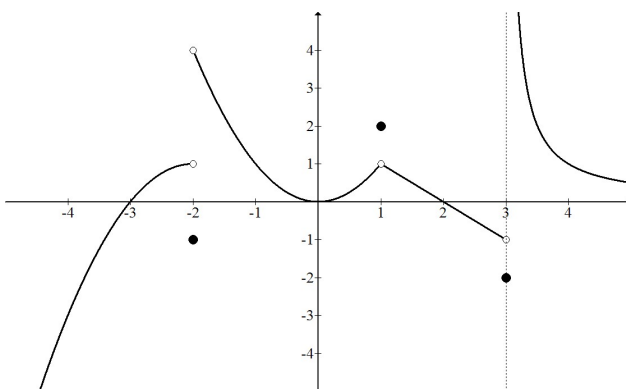


We see from the graph, the limit is  $-\infty$ .

**Homework**

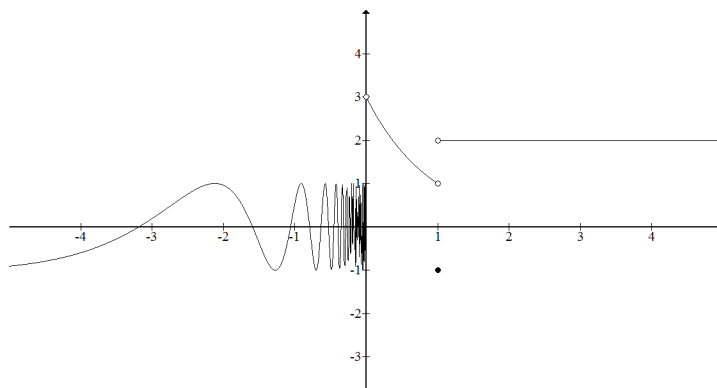
1. For the function graphed below, state the following. If a limit does not exist, write DNE.

- |                                      |                                     |                                     |
|--------------------------------------|-------------------------------------|-------------------------------------|
| (a) $\lim_{x \rightarrow -2^-} f(x)$ | (e) $\lim_{x \rightarrow 1^-} f(x)$ | (i) $\lim_{x \rightarrow 3^-} f(x)$ |
| (b) $\lim_{x \rightarrow -2^+} f(x)$ | (f) $\lim_{x \rightarrow 1^+} f(x)$ | (j) $\lim_{x \rightarrow 3^+} f(x)$ |
| (c) $\lim_{x \rightarrow -2} f(x)$   | (g) $\lim_{x \rightarrow 1} f(x)$   | (k) $\lim_{x \rightarrow 3} f(x)$   |
| (d) $f(-2)$                          | (h) $f(1)$                          | (l) $f(3)$                          |



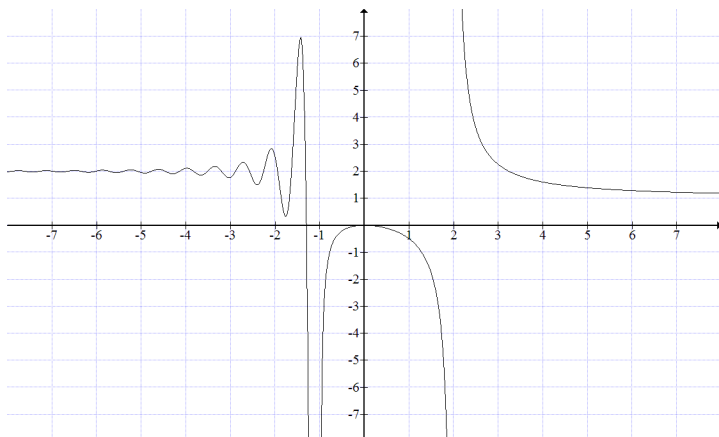
2. For the function graphed below, state the following. If a limit does not exist, write DNE.

- |                                     |                                     |                                     |            |
|-------------------------------------|-------------------------------------|-------------------------------------|------------|
| (a) $\lim_{x \rightarrow 0^-} f(x)$ | (c) $\lim_{x \rightarrow 0} f(x)$   | (e) $\lim_{x \rightarrow 1^+} f(x)$ | (g) $f(1)$ |
| (b) $\lim_{x \rightarrow 0^+} f(x)$ | (d) $\lim_{x \rightarrow 1^-} f(x)$ | (f) $\lim_{x \rightarrow 1} f(x)$   |            |

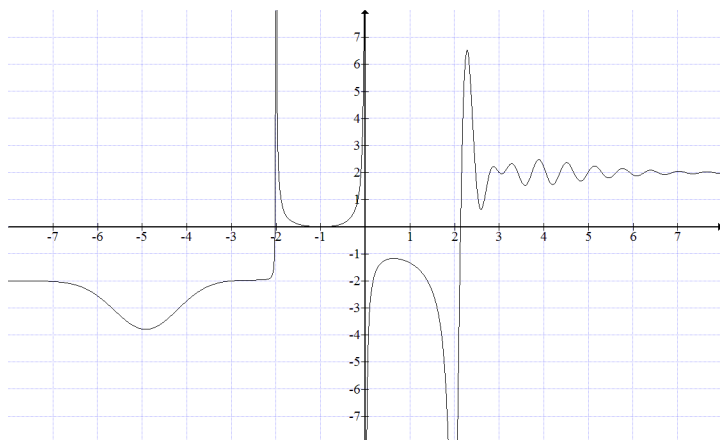


For the functions graphed below, state the following. If a limit does not exist, write DNE.

3. (a)  $\lim_{x \rightarrow -1^-} f(x)$  (c)  $\lim_{x \rightarrow -1} f(x)$  (e)  $\lim_{x \rightarrow 2^+} f(x)$   
 (b)  $\lim_{x \rightarrow -1^+} f(x)$  (d)  $\lim_{x \rightarrow 2^-} f(x)$  (f)  $\lim_{x \rightarrow 2} f(x)$



4. (a)  $\lim_{x \rightarrow -2^-} f(x)$  (d)  $\lim_{x \rightarrow 0^-} f(x)$  (g)  $\lim_{x \rightarrow 2^-} f(x)$   
 (b)  $\lim_{x \rightarrow -2^+} f(x)$  (e)  $\lim_{x \rightarrow 0^+} f(x)$  (h)  $\lim_{x \rightarrow 2^+} f(x)$   
 (c)  $\lim_{x \rightarrow -2} f(x)$  (f)  $\lim_{x \rightarrow 0} f(x)$  (i)  $\lim_{x \rightarrow 2} f(x)$



5. Sketch the graph of

$$f(x) = \begin{cases} 2x & \text{if } x < 2 \\ x^2 & \text{if } x \geq 2 \end{cases}$$

and identify each limit.

- (a)  $\lim_{x \rightarrow 2^-} f(x)$
- (b)  $\lim_{x \rightarrow 2^+} f(x)$
- (c)  $\lim_{x \rightarrow 2} f(x)$
- (d)  $\lim_{x \rightarrow 1} f(x)$

6. Sketch the graph of

$$f(x) = \begin{cases} x^3 - 1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \sqrt{x+1} - 2 & \text{if } x > 0 \end{cases}$$

and identify each limit.

- (a)  $\lim_{x \rightarrow 0^-} f(x)$
- (b)  $\lim_{x \rightarrow 0^+} f(x)$
- (c)  $\lim_{x \rightarrow 0} f(x)$
- (d)  $\lim_{x \rightarrow -1} f(x)$
- (e)  $\lim_{x \rightarrow 3} f(x)$

7. Sketch the graph of

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < -1 \\ 3x + 1 & \text{if } x \geq -1 \end{cases}$$

and identify each limit.

- (a)  $\lim_{x \rightarrow -1^-} f(x)$
- (b)  $\lim_{x \rightarrow -1^+} f(x)$
- (c)  $\lim_{x \rightarrow -1} f(x)$
- (d)  $\lim_{x \rightarrow 1} f(x)$

8. Sketch the graph of

$$f(x) = \begin{cases} 2x + 1 & \text{if } x < -1 \\ 3 & \text{if } -1 \leq x \leq 1 \\ 2x + 1 & \text{if } 1 < x \end{cases}$$

and identify each limit.

- (a)  $\lim_{x \rightarrow -1^-} f(x)$
- (b)  $\lim_{x \rightarrow -1^+} f(x)$
- (c)  $\lim_{x \rightarrow -1} f(x)$
- (d)  $\lim_{x \rightarrow 1} f(x)$
- (e)  $\lim_{x \rightarrow 0} f(x)$

9. Sketch a graph of a function with the given properties:

$$f(-1) = 2, \quad f(0) = -1, \quad f(1) = 3$$

and  $\lim_{x \rightarrow 1} f(x)$  does not exist.

10. Sketch a graph of a function with the given properties:

$$f(x) = 1 \text{ for } -2 \leq x \leq 1,$$

$$\lim_{x \rightarrow 1^+} f(x) = 3 \text{ and } \lim_{x \rightarrow -2} f(x) = 1.$$

11. Sketch a graph of a function with the given properties:

$$f(0) = 1, \quad \lim_{x \rightarrow 0^-} f(x) = 2$$

and  $\lim_{x \rightarrow 0^+} f(x) = 3.$

12. Sketch a graph of a function with the given properties:

$$\lim_{x \rightarrow 0} f(x) = -2, \quad f(0) = 1,$$

$$f(2) = 3,$$

and  $\lim_{x \rightarrow 2} f(x)$  does not exist.

Determine each limit. Your answer should either be a number,  $\infty$ ,  $-\infty$ , or does not exist.

13.  $\lim_{x \rightarrow 1^-} \frac{x}{x-1}$

14.  $\lim_{x \rightarrow -5^+} \frac{6}{x+5}$

15.  $\lim_{x \rightarrow 1^+} \frac{1-2x}{x^2-1}$

16.  $\lim_{x \rightarrow 2^-} \frac{x-4}{x^2-4x+4}$

17.  $\lim_{x \rightarrow 2^-} \frac{-x}{\sqrt{4-x^2}}$

18.  $\lim_{x \rightarrow -2^-} \frac{x}{x^2-4}$

**Homework Solutions**

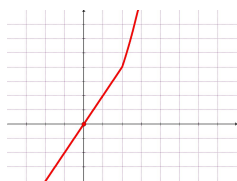
1. (a) 1 (b) 4 (c) DNE (d) -1 (e) 1 (f) 1 (g) 1 (h) 2 (i) -1 (j) DNE (k) DNE (l) -2

2. (a) DNE (b) 3 (c) DNE (d) 1 (e) 2 (f) DNE (g) -1

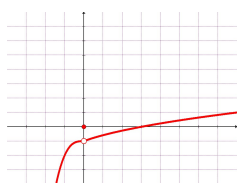
3. (a)  $-\infty$  (b)  $-\infty$  (c)  $-\infty$  (d)  $-\infty$  (e)  $\infty$  (f) DNE

4. (a)  $\infty$  (b)  $\infty$  (c)  $\infty$  (d)  $\infty$  (e)  $-\infty$  (f) DNE (g)  $-\infty$  (h)  $-\infty$  (i)  $-\infty$

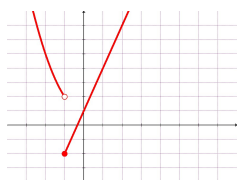
5. (a) 4 (b) 4 (c) 4 (d) 2



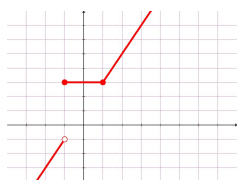
6. (a) -1 (b) -1 (c) -1 (d) -2 (e) 0



7. (a) 2 (b) -2 (c) DNE (d) 4



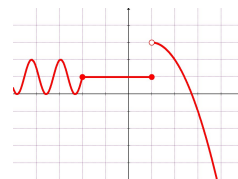
8. (a) -1 (b) 3 (c) DNE (d) 3 (e) 3



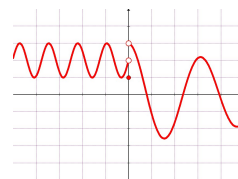
9. .



10. .



11. .



12. .



13.  $-\infty$

14.  $\infty$

15.  $-\infty$

16.  $-\infty$

17.  $-\infty$

18.  $-\infty$