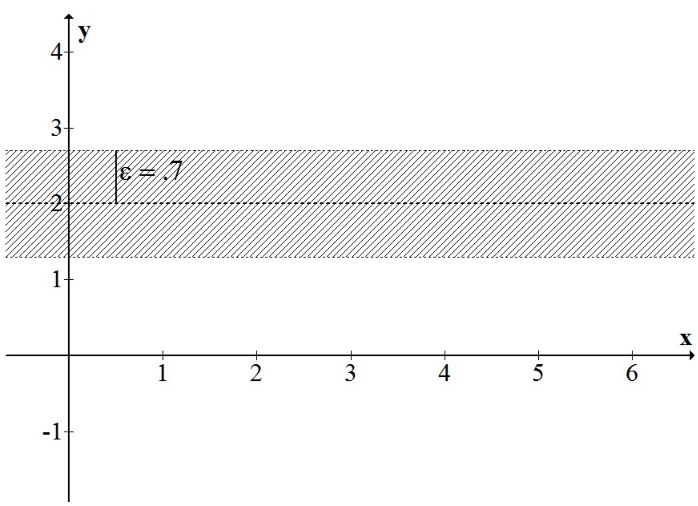
2 Limits and Derivatives

2.4 The Precise Definition of Limit

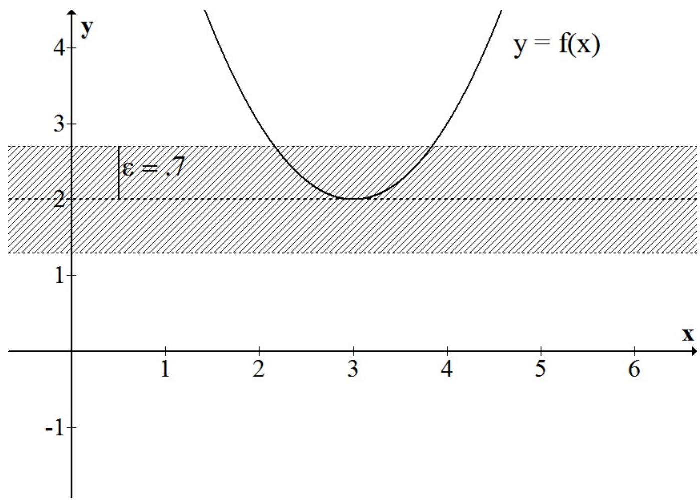
The lower case Greek letter ε is epsilon. The lower case Greek letter δ is delta.

The figure below shows an ε-band (epsilon band) centered around the line $y = 2$ with radius $\varepsilon = 0.7$. A point (x, y) is within the epsilon band if and only if the distance between y and 2 is less than $\varepsilon = 0.7$.

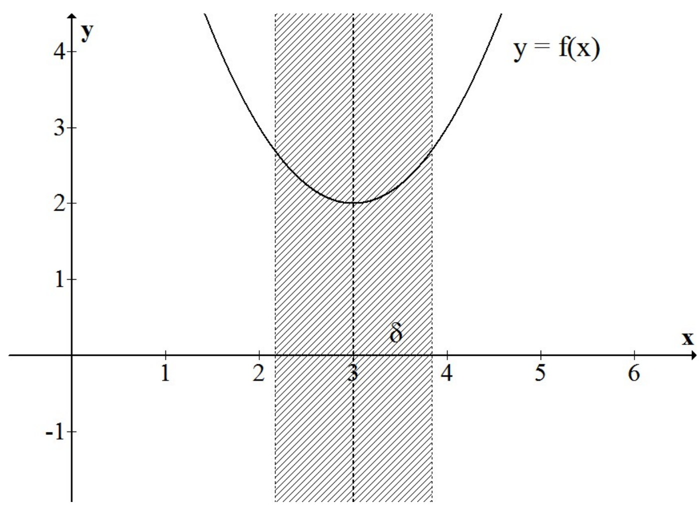
The distance between two numbers is found taking the difference of the two numbers and then taking the absolute value of that difference, just in case the result is negative. The distance between y and 2 is equal to $|y - 2|$. Therefore, the point (x, y) is in the epsilon band if and only if $|y - 2| < 0.7$.



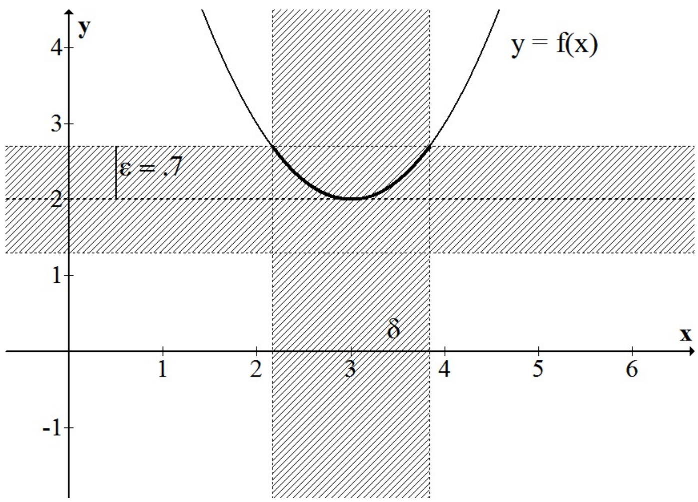
If the limit of a function $f(x)$ as x approaches 3 is the y-value 2, that is $\lim_{x \to 3} f(x) = 2$, then eventually the graph of f will be within the ε-band as x approaches 3. That is, $|f(x) - 2| < 0.7$ whenever x is sufficiently close to 3.



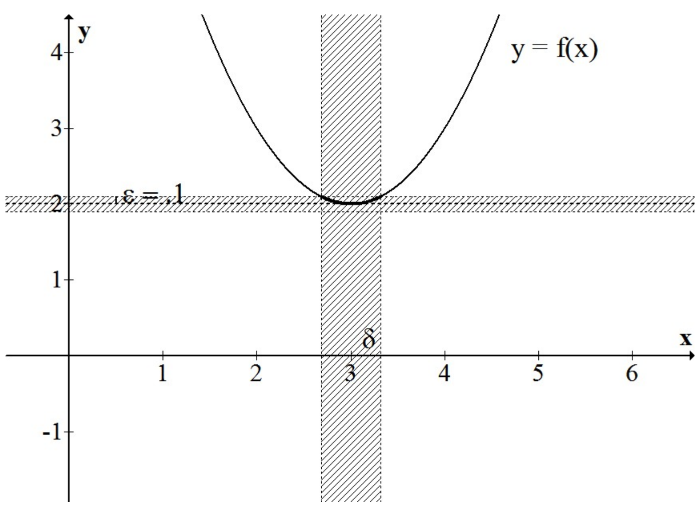
We can make a δ-band (delta band) about the number $x = 3$. A point (x, y) is within the delta band if and only if the distance between x and 3 is less than δ. That is, $|x - 3| < \delta$.



Suppose that $\lim_{x \to 3} f(x) = 2$. We should be able to make a δ-band thin enough so that if the graph of f is inside the δ-band, then the graph of f is inside the ε-band. It should not matter if we overdo it and make the δ-band thinner than is needed.



If the limit is 2, we should be able to make the ε-band as thin as we want and still be able find a δ-band such that if the graph of f is inside the δ-band, then the graph of f will be inside the ε-band. It should not matter how big epsilon is. Below we pick $\varepsilon = 0.1$.



In this section, a precise definition of limit will be given.

Definition of Limit The limit of $f(x)$ as x approaches a equals the number L, that is,

$$\lim_{x \to a} f(x) = L \text{ if and only if}$$

for every ε-band around $y = L$, no matter how thin, we can make a δ-band around $x = a$, such that if the graph of f lies in the delta band (but x does not equal a), then the graph of f will lie in the epsilon band.

We next write an equivalent definition. Note that the graph of f lies in an epsilon band around $x = a$ if and only if $|x - a| < \delta$. Also note that the graph of f lies in an epsilon band around $y = L$ if and only if $|f(x) - L| < \varepsilon$.

A Precise Definition of Limit Let f be a function defined on some open interval containing the number a, except possibly a itself. Then

$$\lim_{x \to a} f(x) = L \text{ if and only if}$$

for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that

$$\text{if } |f(x) - L| < \varepsilon \text{ then } 0 < |x - a| < \delta.$$

Note that the definition requires that $0 < |x - a| < \delta$. The reason $|x - a|$ must be greater than zero is because x should not equal a. The value of the limit has to do with what happens to f as x gets very close to a, but not equal to a.

Consider the function $f(x) = 2x + 1$. Suppose we want to calculate the limit of $f(x)$ as x approaches 3. We can calculate the limit by direct substitution because f is a polynomial. Therefore, the limit is $f(3) = 7$ We have

$$\lim_{x \to 3} (2x + 1) = 2(3) + 1 = 7$$

According to the precise definition, if we arbitrarily choose a positive number ε then there must exist a positive number δ such that if $0 < |x - 3| < \delta$ then $|f(x) - 7| < \varepsilon$. Note that $L = 7$ and $a = 3$.

What should δ be when $\varepsilon = 1/10$? The answer is found by doing the the following calculation.

$$\begin{aligned} |f(x) - L| &< \varepsilon \\ |(2x+1) - 7| &< 1/10 \\ |2x - 6| &< 1/10 \\ 2|(x-3)| &< 1/10 \\ |(x-3)| &< 1/20 \end{aligned}$$

Therefore, we can choose $\delta = 1/20$. We have $|(2x+1) - 7| < \varepsilon$ whenever $0 < |x - 3| < \delta$.

For the last computation we used the following property of absolute values.

Property $|ab| = |a| \cdot |b|$

Example 1 If $\lim_{x\to 3}(2x+1) =$, find a value for δ when

1. $\varepsilon = 1/100$
2. $\varepsilon = 1/1000$

SOLUTION

1. $\varepsilon = 1/100$

$$|f(x) - L| < \varepsilon$$
$$|(2x+1) - 7| < 1/100$$
$$|2x - 6| < 1/100$$
$$2|(x-3)| < 1/100$$
$$|(x-3)| < 1/200$$

2. $\varepsilon = 1/1000$

$$
\begin{aligned}
&|f(x) - L| < \varepsilon \\
&|(2x+1) - 7| < 1/1000 \\
&|2x - 6| < 1/1000 \\
&2|(x-3)| < 1/1000 \\
&|(x-3)| < 1/2000 \qquad \square
\end{aligned}
$$

We see that δ should always be half of ε. This is not a general rule for all functions. It is just the case for the function that we are studying here at the particular value of $a = 3$.

We now would like to do the same type of calculation, but in a more general way.

Example 2 Given that $\lim_{x \to 4}(5x - 2) = 18$, find a $\delta > 0$ that is a function of ε that is guaranteed to exist under the definition of limit.

SOLUTION

$$
\begin{aligned}
|f(x) - L| &< \varepsilon \\
|(5x-2) - 18| &< \varepsilon \\
|5x - 20| &< \varepsilon \\
|5(x-4)| &< \varepsilon \\
5|x-4| &< \varepsilon \\
|x-4| &< \varepsilon/5
\end{aligned}
$$

Therefore, we can choose $\delta = \varepsilon/5$.

Using the precise definition of limit to prove that a function has Limit L as x approaches a number a.

Example 3 Use the precise definition of limit to prove that

$$\lim_{x \to 5}(7x - 2) = 33.$$

We must find a δ that is a function ε which is must exist under the definition of limit. We follow the steps of the previous example.

$$\begin{aligned} |f(x) - L| &< \varepsilon \\ |(7x-2) - 33| &< \varepsilon \\ |7x - 35| &< \varepsilon \\ |7(x-5)| &< \varepsilon \\ 7|x-5| &< \varepsilon \\ |x-5| &< \varepsilon/7 \end{aligned}$$

Therefore, we can choose $\delta = \varepsilon/7$. We can now write a formal proof. The proof is written in a format that should be followed when doing the homework.

$$\lim_{x \to 5}(7x - 2) = 33.$$

PROOF:

Let $\varepsilon > 0$ and choose $\delta = \varepsilon/7$.

If $0 < |x - 5| < \delta$ then

$$|f(x)-L| = |(7x-2)-33| = |7x-35| = |7(x-5)| = 7|x-5| < 7\delta = 7\cdot\left(\frac{\varepsilon}{7}\right) = \varepsilon$$

That is, if $0 < |x - 5| < \delta$, then $|(7x - 2) - 33| < \varepsilon$ □

Example 4 Use the precise definition of limit to prove that

$$\lim_{x \to -2}(-3x + 1) = 7.$$

Find δ.

$$\begin{aligned} |f(x) - L| &< \varepsilon \\ |(-3x+1) - 7| &< \varepsilon \end{aligned}$$

$$\begin{aligned} |-3x-6| &< \varepsilon \\ |-3(x+2)| &< \varepsilon \\ |-3|\cdot|x-(-2)| &< \varepsilon \\ |x-(-2)| &< \varepsilon/3 \end{aligned}$$

Therefore, we can choose $\delta = \varepsilon/3$. We can now write a formal proof.

$$\lim_{x\to -2}(-3x+1) = 7.$$

PROOF:
Let $\varepsilon > 0$ and choose $\delta = \varepsilon/3$.

If $0 < |x-(-2)| < \delta$ then

$$|f(x)-L| = |(-3x+1)-7| = |-3x-6| = |-3(x+2)| = |-3|\cdot|x-(-2)| < 3\delta = 3\cdot\left(\frac{\varepsilon}{3}\right) = \varepsilon$$

That is, if $0 < |x-(-2)| < \delta$, then $|(-3x+1)-7| < \varepsilon$ $\square$

In the last section we used the Limit Laws to calculate the limits of functions. However, the Limit Laws were given without proof. To prove the Limit Laws we need to use the precise definition of limit. This is usually a topic for an advanced calculus class.

In the examples shown here, the functions have been linear functions. The problems can be more difficult with other functions.

Homework

For these problems, assume that $\lim_{x\to a} f(x) = L$. Find δ as a function of ϵ guaranteed to exist under the definition of limit.

1. $\lim_{x\to 2}(3x+1) = 7, \quad f(x) = 3x+1,\ a = 2,\ L = 7,\ \varepsilon = 0.1$
2. $\lim_{x\to 5}(7x+1) = 36, \quad f(x) = 7x+1,\ a = 5,\ L = 36,\ \varepsilon = 0.1$
3. $\lim_{x\to 1}(-2x+1) = -1,\ \varepsilon = 0.1$
4. $\lim_{x\to -2}(3x+1) = -5,\ \varepsilon = 0.1$

   Find δ as a function of ϵ guaranteed to exist under the definition of limit. Then prove the statement using the precise definition of limit.

5. $\lim_{x\to 2}(3x+1) = 7$
6. $\lim_{x\to 5}(7x+1) = 36$
7. $\lim_{x\to 1}(-2x+1) = -1$
8. $\lim_{x\to -2}(3x+1) = -5$

Solution The solutions for the problems 5 through 8 should include proofs. I did not write the proofs here. Let's go over the proofs in class!

1. $\delta = 0.1/3$
2. $\delta = 0.1/7$
3. $\delta = 0.1/2$
4. $\delta = 0.1/3$
5. $\delta = \varepsilon/3$
6. $\delta = \varepsilon/7$
7. $\delta = \varepsilon/2$
8. $\delta = \varepsilon/3$