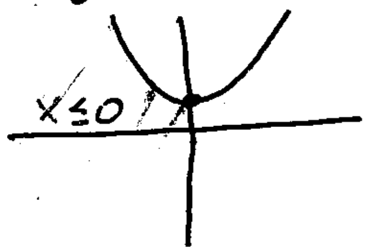
Math 180 Weds. Feb. 10, 2010

§ 2.5 # 37

$$f(x) = \begin{cases} 1 + x^2 & \text{if } x \le 0 \\ 2 - x & \text{if } 0 < x \le 2 \\ (x-2)^2 & \text{if } x > 2 \end{cases}$$

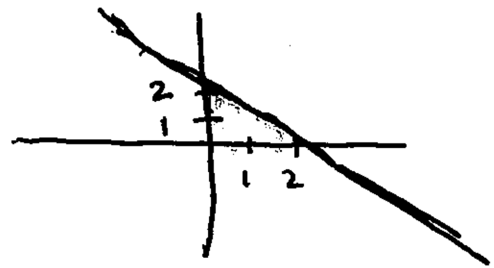
Optional Thumbnail sketches

- $y = x^2 + 1$



- $y = -x + 2$ $m = -1$, $b = 2$

x	y
0	2
2	0

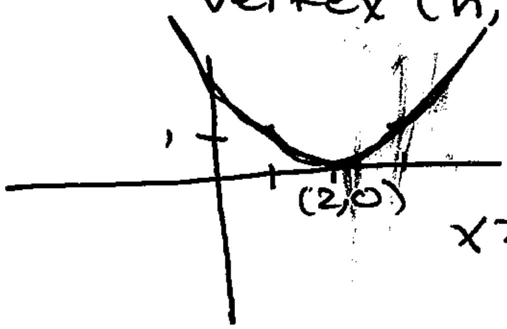


$0 < x \le 2$

- $y = (x-2)^2$

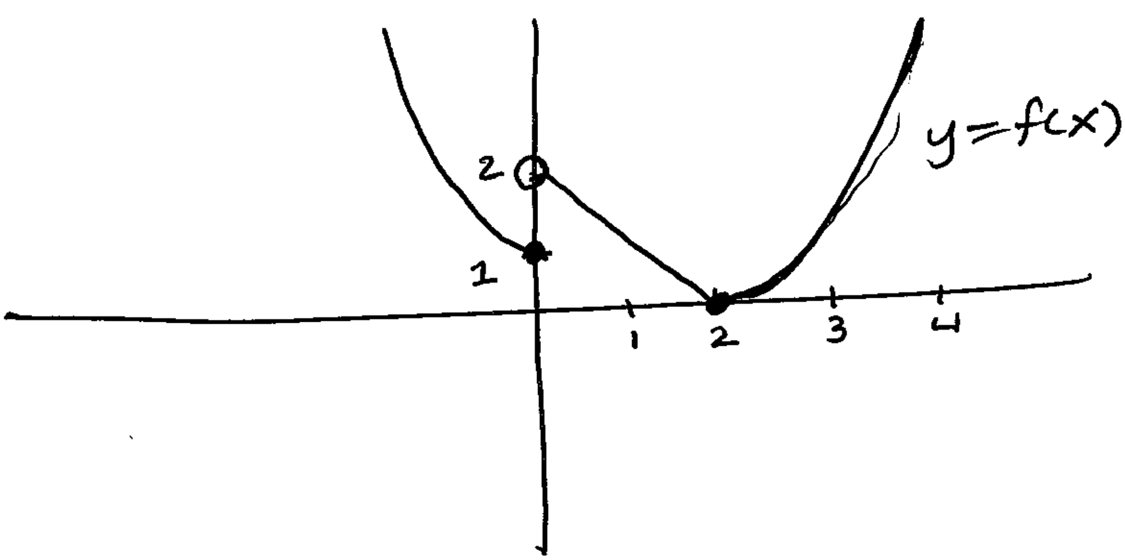
vertex $(2, 0)$

$y = a(x-h)^2 + k$
vertex (h, k)



$x > 2$

Put it all together:



Where is f discontinuous?
Which requirement from the three part definition of continuity fails?

Answer:

$x = 0$

because

$\lim_{x \to 0} f(x)$ DNE

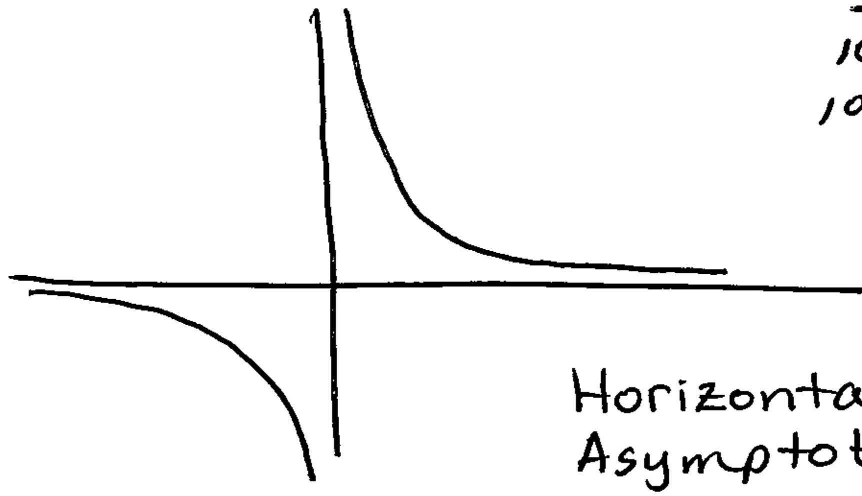
Aside: State the three part definition of a function f continuous at a number a.

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

§2.6 Limits at Infinity; Horizontal Asymptot

HW §2.6 # 3-10, 15-36, 39-44

Let's look at the graph of
$y = \frac{1}{x}$



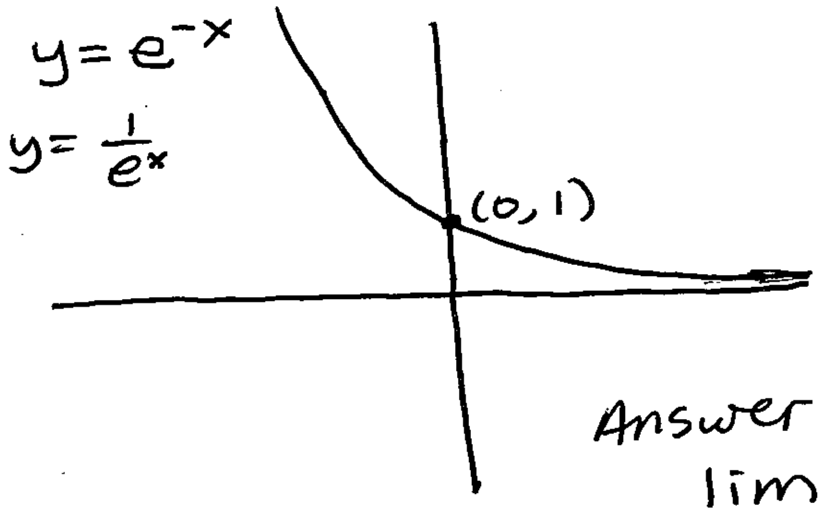
x	$y = 1/x$
10	$1/10$
100	$\frac{1}{100}$

$\frac{1}{\text{big}} = \text{small}$

Horizontal Asymptote: The x-axis.

We write
$$\lim_{x \to \infty} \frac{1}{x} = 0$$
and
$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

<u>Example</u>: Sketch the graph of $y=e^{-x}$. Then evaluate $\lim_{x\to\infty} e^{-x}$.



Answer:

$$\lim_{x\to\infty} e^{-x} = 0$$

Horizontal Asymptote: The x-axis. (The line $y=0$)

<u>Theorem</u>: If $r>0$ is a rational number, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0$$

and $$\lim_{x\to-\infty} \frac{1}{x^r} = 0$$

provided negative numbers are in the domain of $\frac{1}{x^r}$.

Example: Evaluate.

① $\lim_{x\to\infty} \frac{1}{x^2} = 0$

② $\lim_{x\to\infty} \frac{1}{x^{10}} = 0$

③ $\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$ ($\sqrt{x}$ ← $x^{1/2}$)

④ $\lim_{x\to\infty} \frac{1}{x^3} = 0$

⑤ $\lim_{x\to\infty} \frac{1}{x^{-2}} = \lim_{x\to\infty} x^2 = \infty$

Example Find the limit.

$$\lim_{x\to\infty} \frac{3x^4+5x^2+1}{x^4+6}$$

$$= \lim_{x\to\infty} \left(\frac{3x^4+5x^2+1}{x^4+6}\right)\frac{1/x^4}{1/x^4}$$

Divide num & denom by highest degree term in the denominator.

$$= \lim_{x\to\infty} \left(\frac{3x^4/x^4 + 5x^2/x^4 + 1/x^4}{x^4/x^4 + 6/x^4}\right)$$

$$= \lim_{x\to\infty} \left(\frac{3 + \overbrace{5/x^2}^{\to 0} + \overbrace{1/x^4}^{\to 0}}{1 + \underbrace{6/x^4}_{\to 0}}\right)$$

The limit laws apply for $x\to\infty$.

$$= \frac{3}{1} = 3$$

If deg num = deg denom. then the limit equals $\frac{\text{lead coeff num}}{\text{lead coeff denom}}$.

Horizontal Asymptote $\boxed{y = 3}$

<u>Example</u>: Find the limit.

$$\lim_{x \to -\infty} \frac{x^2+3}{x^3+5x^2+2x+7}$$

$$= \lim_{x \to -\infty} \left(\frac{x^2+3}{x^3+5x^2+2x+7} \right) \frac{1/x^3}{1/x^3}$$

$$= \lim_{x \to -\infty} \frac{\overset{\to 0}{1/x} + \overset{\to 0}{3/x^3}}{\underset{\downarrow\, 1}{1} + \underset{\downarrow\, 0}{5/x} + \underset{\downarrow\, 0}{2/x^2} + \underset{\downarrow\, 0}{7/x^3}} = \frac{0}{1} = 0$$

Horizontal Asymptote $y=0$, (x-axis)

<u>Rule of Thumb</u>: If deg num < deg denom then the limit is 0.

Example

$$\lim_{x \to \infty} \frac{2x+3}{\sqrt{9x^2+1}}$$

$$= \lim_{x \to \infty} \frac{(2x+3)\cdot \frac{1}{x}}{\left(\sqrt{9x^2+1}\right)\frac{1}{x}} \qquad \frac{1}{x} = \frac{1}{\sqrt{x^2}}$$

$$= \lim_{x \to \infty} \frac{2 + 3/x}{\sqrt{(9x^2+1)\,1/x^2}}$$

$$= \lim_{x \to \infty} \frac{2 + \underbrace{3/x}_{\to 0}}{\sqrt{9 + \underbrace{1/x^2}_{\to 0}}}$$

$$= \frac{2}{\sqrt{9}} = \frac{2}{3}$$

$\sqrt{x^2} = x$ if $x \geq 0$

$\sqrt{x^2} = -x$ if $x < 0$

$\Downarrow$

$\sqrt{(-5)^2} = \sqrt{25} = 5 = -(-5)$

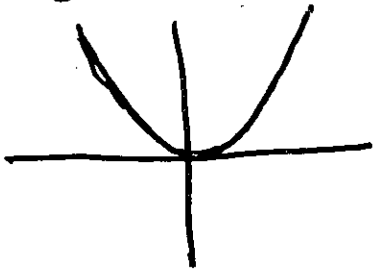
in general

$\sqrt{x^2} = |x|$

The Behavior of Polynomials as $x \to \infty$.

Gallery:

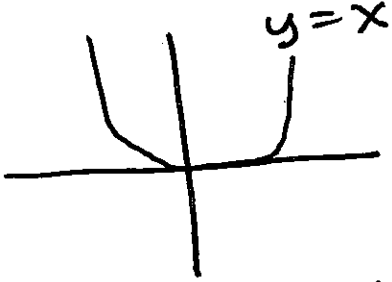
$y = x^2$



$$\lim_{x \to \infty} x^2 = \infty$$

$$\lim_{x \to -\infty} x^2 = \infty$$

$y = x^4$



$$\lim_{x \to \infty} x^4 = \infty$$

$$\lim_{x \to -\infty} x^4 = \infty$$

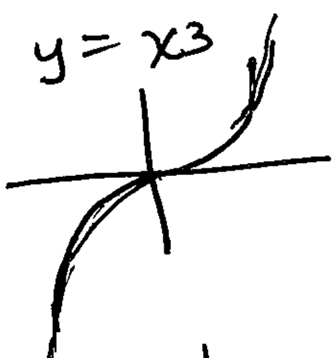
$y = x^n$

n is even

$$\lim_{x \to \infty} x^n = \infty$$

$$\lim_{x \to -\infty} x^n = \infty$$

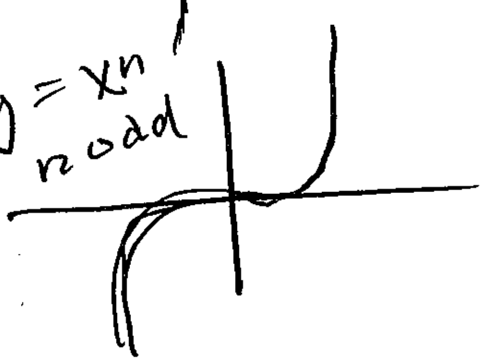
$y = x^3$



$$\lim_{x \to \infty} x^3 = \infty$$

$$\lim_{x \to -\infty} x^3 = -\infty$$

$y = x^n$
n odd



If n is odd

$$\lim_{x \to \infty} x^n = \infty$$

$$\lim_{n \to -\infty} x^n = -\infty$$

FACT: As $x \to \infty$, polynomials behave like their highest degree term.

Example: Find the infinite limit.

① $\lim_{x\to\infty} x^4 - 1{,}00x^3 - 1{,}000x^2 - 10{,}000$

$= \lim_{x\to\infty} x^4 = \infty$

② $\lim_{x\to\infty} -x^7 + 5x^2 + 3$

$= \lim_{x\to\infty} -x^7 = -\infty$

Explanation:

$\lim_{x\to\infty} (-x^7 + 5x^2 + 3)$

$$= \lim_{x\to\infty} -x^7 \Big(\underbrace{1 - \overset{\to 0}{\frac{5}{x^5}} + \overset{\to 0}{\frac{3}{x^7}}}_{\to 1}\Big)$$

$= \lim_{x\to\infty} (-x^7) = -\infty$

§ 2.5 #16

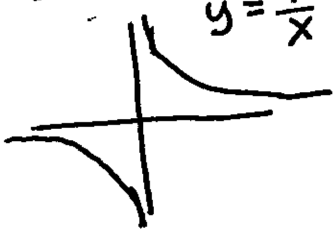
- State the three part definition for a function f to be continuous at a number.

  1. $f(a)$ is defined
  2. $\lim_{x \to a} f(x)$ exists
  3. $\lim_{x \to a} f(x) = f(a)$

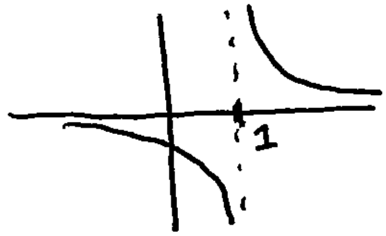
- Let $f(x) = \begin{cases} \frac{1}{x-1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$

  Sketch the graph.

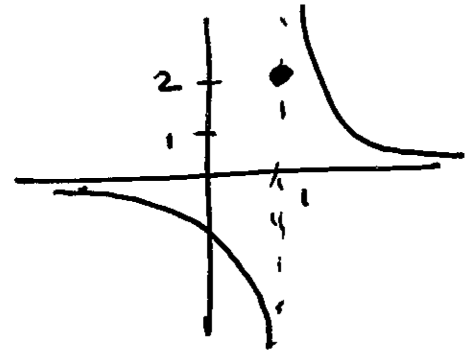
<u>Aside</u> $y = \frac{1}{x}$



$y = \frac{1}{(x-1)}$



<u>$y = f(x)$</u>



- For which value of x is f discontinuous? State which requirement fails.

<u>SOLUTION</u> $a = 1$

because $\lim_{x \to 1} f(x)$ DNE.

§2.5 #20

- State the three part def. of f contin. at a.
  1. $f(a)$ is defined
  2. $\lim_{x \to a} f(x)$ exists
  3. $\lim_{x \to a} f(x) = f(a)$

- Let $f(x) = \begin{cases} \dfrac{2x^2-5x-3}{x-3} & \text{if } x \neq 3 \\ 6 & \text{if } x=3 \end{cases}$

Aside

$$y = \frac{2x^2-5x-3}{x-3}$$

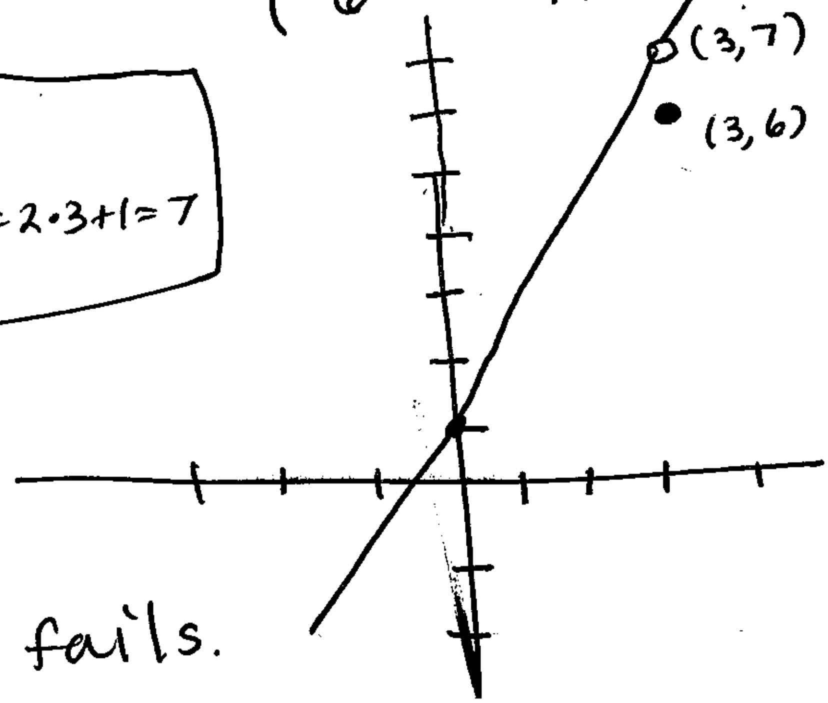
$$= \frac{\cancel{(x-3)}(2x+1)}{\cancel{(x-3)}}$$

$= 2x+1$ when $x \neq 3$

Line with a hole at $x=3$, $y = 2\cdot 3+1 = 7$

$(3,7)$.

$$= \begin{cases} 2x+1 & \text{if } x \neq 3 \\ 6 & \text{if } x=3 \end{cases}$$



- State the value of x where f is discontinuous. State which requirement fails.

$x = 3$

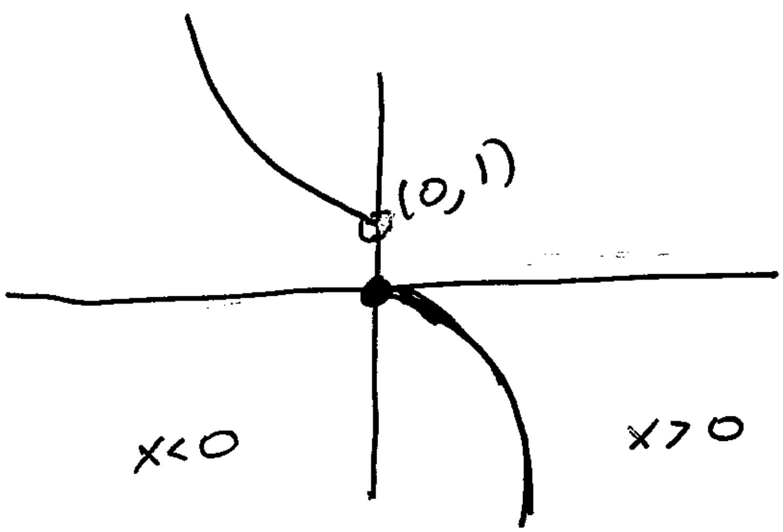
because $\lim_{x \to 3} f(x) \neq f(3)$

§2.5 Practice

- State the three part definition for a function f to be continuous at a number.

1. $f(a)$ is defined (exists)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

- Sketch the graph. $f(x) = \begin{cases} -x^2 & \text{if } x \geq 0 \\ e^{-x} & \text{if } x < 0 \end{cases}$

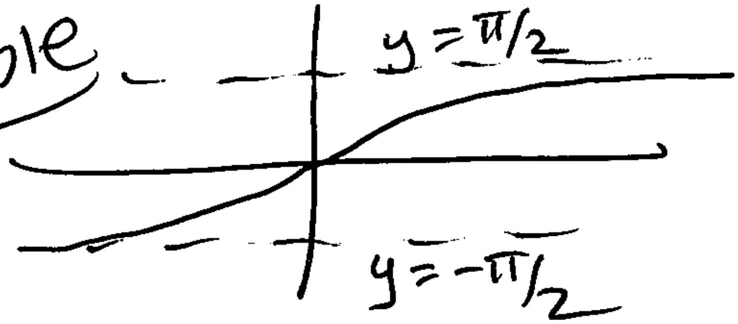


- State the values of x where f is discontinuous. State which requirement fails.

$x = 0$ because $\lim_{x \to 0} f(x)$ DNE.

Aside • $y = \tan^{-1} x$

Example



• $\lim\limits_{x \to \infty} \tan^{-1} x = \pi/2$

• $\lim\limits_{x \to -\infty} \tan^{-1} x = -\pi/2$

Horizontal Asymptotes

$y = \pi/2, \; y = -\pi/2$