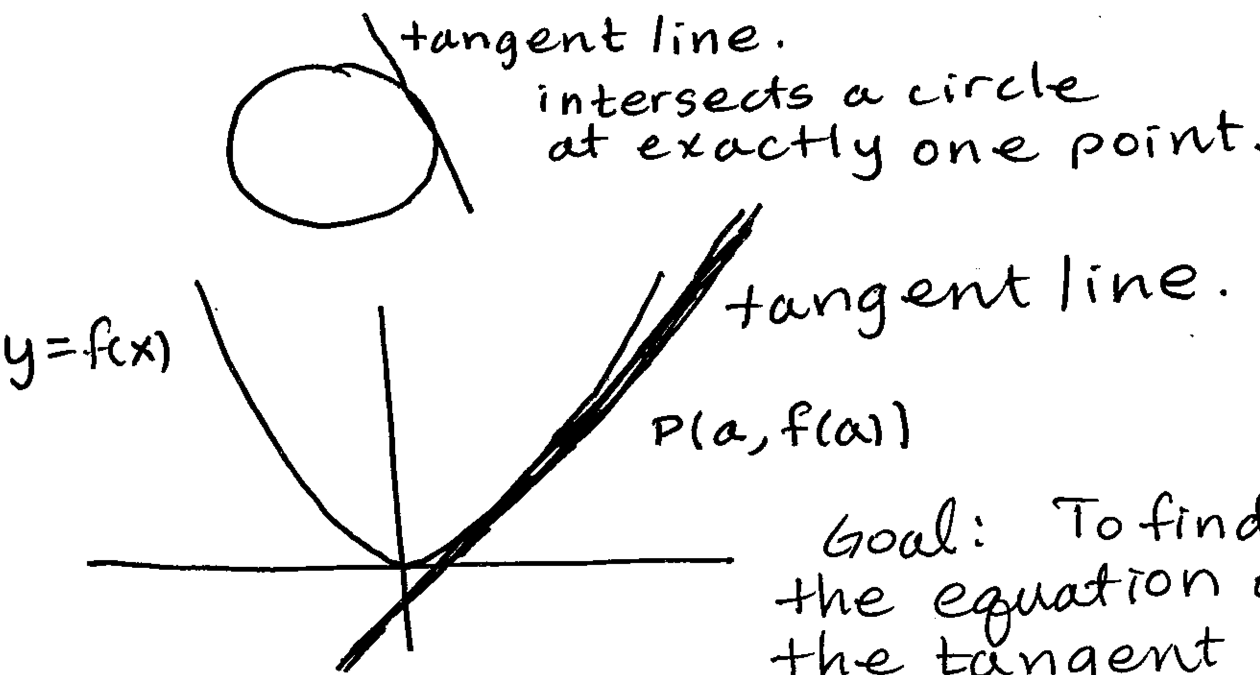
§2.7 Derivatives and Rates of Change

§2.7 HW # 5, 6, 7, 8

Next Thurs. Feb. 18
Test §2.2, 2.3, 2.4, 2.5, 2.6, 2.7
That is §2.2–2.7.

The Tangent Line to a Curve



Goal: To find the equation of the tangent line.

Point-slope Equation of a Line

$$y - y_1 = m(x - x_1)$$

(x_1, y_1) is a point on the line
m is the slope

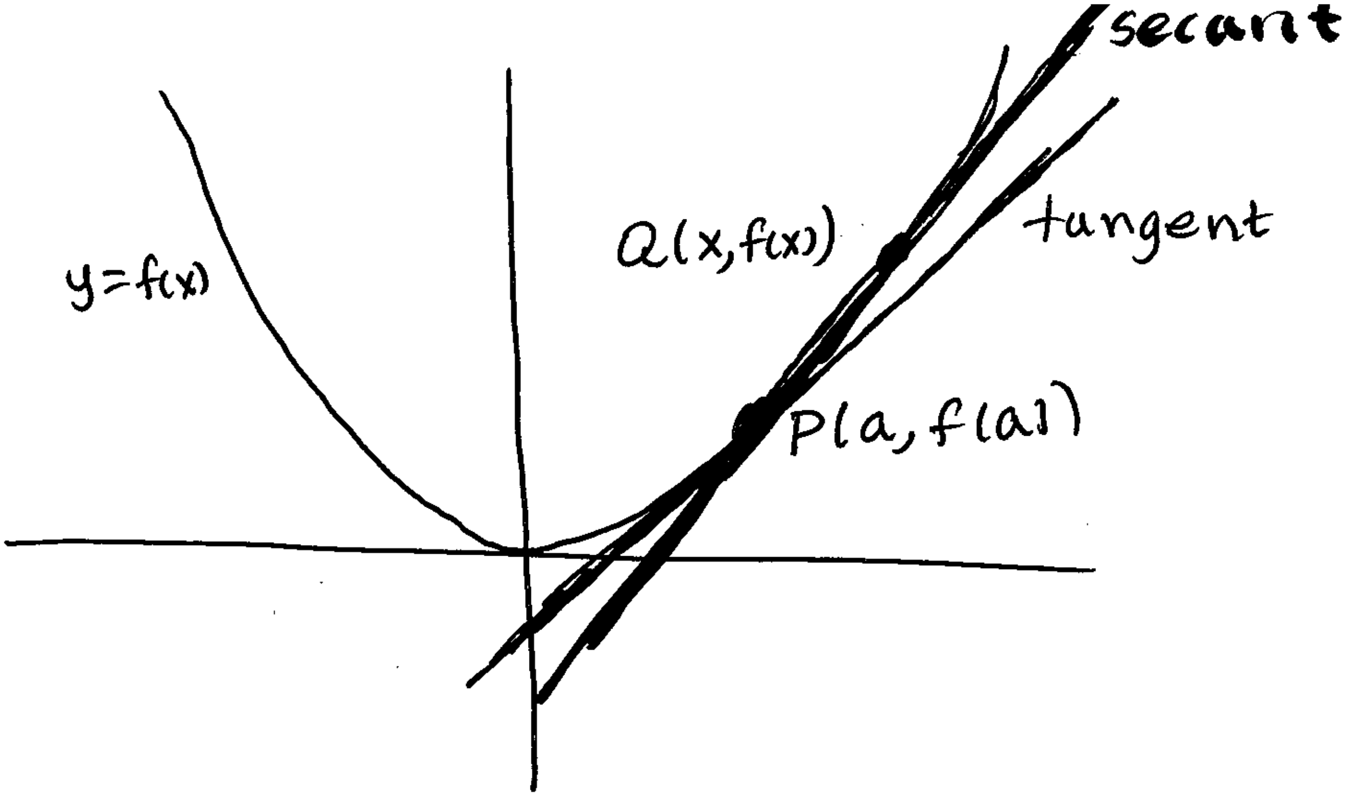
Example: Find the equation of the line that passes through the points $(1, 3)$ and $(5, 11)$.

SOLUTION. Find m

$$m = \frac{\text{rise}}{\text{run}} = \frac{11-3}{5-1} = \frac{8}{4} = 2$$

$x_1 = 1$, $y_1 = 3$, $m = 2$

$$y - y_1 = m(x - x_1)$$
$$y - 3 = 2(x - 1)$$
$$y - 3 = 2x - 2$$
$$y = 2x - 2 + 3$$
$$\boxed{y = 2x + 1}$$



To find the slope of a line, typically we need two points on the line.
We are only given one point.
We pick another point on the curve f. Call it Q, where Q is near P.

We use the slope of the line PQ (call it the secant line) to approximate the slope of the tangent.

Let's find the slope of the secant line, m_{sec}.

$$m_{sec} = \frac{\text{rise}}{\text{run}} = \frac{f(x) - f(a)}{x - a}$$

We let Q approach P to get a really good ~~approach~~ approximation. Let's make Q infinitely close to P.

We define the slope of the tangent line, m_{tan}, as

$$m_{tan} = \lim_{\substack{\cancel{P \to t} \\ Q \to P}} m_{sec}$$

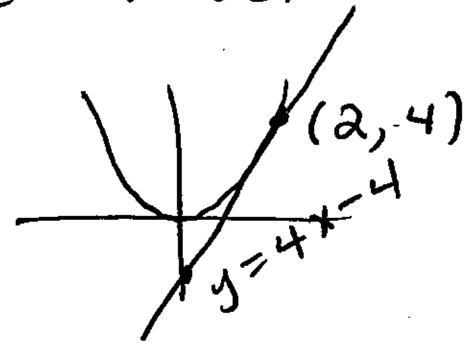
$$\boxed{m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

This is so important, that we call this the <u>derivative</u> of f at a. We write

$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

So derivitive = slope of tangent line.

Example: Let $f(x) = x^2$.
Find the equation of the tangent line to f at $a = 2$.



SOLUTION

We have the point
$(2, 4)$, $a = 2$
not $f(2) = 2^2 = 4$

We need the slope.

$$m_{tan} = f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \to 2} \frac{x^2 - (2)^2}{x - 2}$$

$$= \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}$$

$$= \lim_{x \to 2} (x+2) = 2 + 2 = 4$$

$m_{tan} = 4$

$\boxed{\text{We see that } f'(2) = 4}$

We have
$x_1 = 2, \; y_1 = 4, \; m = 4$

$y - y_1 = m(x - x_1)$

$y - 4 = 4(x - 2)$

$y - 4 = 4x - 8$

$\boxed{y = 4x - 4}$

Example: Find the equation of the tangent line to $f(x) = \sqrt{x}$ ~~for a =~~ ~~at (~~ at $a = 4$.

SOLUTION

$$y - y_1 = m(x - x_1)$$
$$x_1 = 4$$
$$y_1 = f(4) = \sqrt{4} = 2$$
$$m = f'(4)$$

$$m = f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
$$= \lim_{x \to 4} \frac{f(x) - f(4)}{x - 4}$$
$$= \lim_{x \to 4} \frac{\sqrt{x} - \sqrt{4}}{x - 4}$$
$$= \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$$
$$= \lim_{x \to 4} \frac{(\sqrt{x} - 2)}{(x - 4)} \frac{(\sqrt{x} + 2)}{(\sqrt{x} + 2)}$$
$$= \lim_{x \to 4} \frac{\cancel{(x-4)}}{\cancel{(x-4)}(\sqrt{x} + 2)} = \lim_{x \to 4} \frac{1}{\sqrt{x} + 2}$$
$$= \frac{1}{\sqrt{4} + 2} = \frac{1}{2 + 2} = \frac{1}{4}$$
$$m = 1/4$$

$y - y_1 = m(x - x_1)$

$x_1 = 4,\ y_1 = 2,\ m = \frac{1}{4}$

$y - 2 = \frac{1}{4}(x - 4)$

$y - 2 = \frac{1}{4}x - 1$

$\boxed{y = \frac{1}{4}x + 1}$