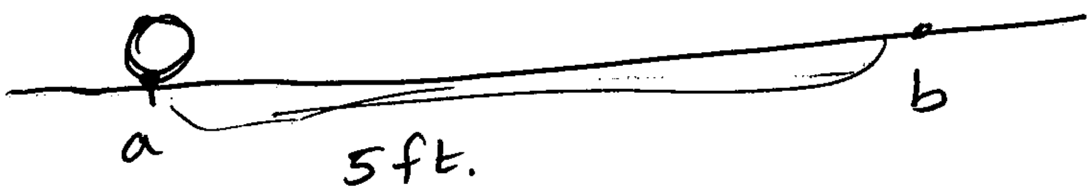
§ 2.7 Continued

Velocity and Rate of Change.

HW § 2.7 #13-15.

$$\text{average velocity} = \frac{\text{distance travelled}}{\cancel{\text{cho}}\ \text{elapsed time}}$$

ball goes from a to b.



It takes 3 sec. for the ball to go from a to b.

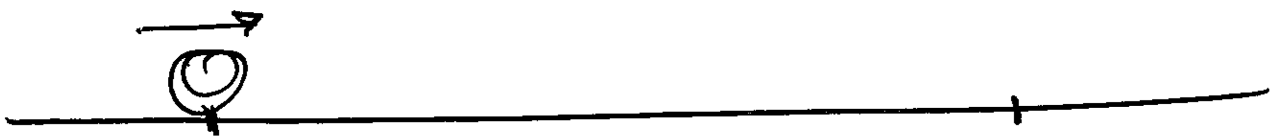
$$V_{AVG} = \frac{\Delta d}{\Delta t} = \frac{5\text{ ft}}{3\text{ sec}} = \frac{5}{3}\text{ ft/s}$$

Δ change

The position of the ball is a function of time.

Let's call the position $s(t)$.

~~If we want~~



~~$t=a$~~

$s(a)$

we start at time $t=a$

$s(t)$

end at time t

$$V_{AVG} = \frac{\Delta \text{dist}}{\Delta \text{time}} = \frac{\Delta s}{\Delta t} = \frac{s(t)-s(a)}{t-a}$$

To find the velocity at time $t=a$, we want our interval to be as small as possible

We let $t \to a$.

Define instantaneous velocity as

$$V(a) = \lim_{t \to a} V_{AVG} = \lim_{t \to a} \frac{s(t)-s(a)}{t-a}$$

We have

$$v(a) = s'(a)$$

velocity = derivative of position function.

$\boxed{\text{Derivative} = \text{Rate of Change}}$

§2.7 HW #14

A rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height (in meters) after t seconds is given by $H(t) = 10t - 1.86t^2$

(a) Find the velocity after ~~1 sec~~ 1 seconds.

$$V(1) = H'(1)$$

$$= \lim_{t \to 1} \frac{H(t) - H(1)}{t-1}$$

$$= \lim_{t \to 1} \frac{(10t - 1.86t^2) - (10(1) - 1.86(1)^2)}{t-1}$$

$$= \lim_{t \to 1} \frac{10t - 1.86t^2 - 8.14}{t-1}$$

$$= \cancel{\lim_{t \to 1} \frac{(t-1)(1.86}{(t-1)}}$$

$$= \lim_{t \to 1} \frac{-1.86t^2 + 10t - 8.14}{t-1}$$

$$= \lim_{t \to 1} \frac{\cancel{(t-1)}(-1.86t + 8.14)}{\cancel{(t-1)}}$$

$$= -1.86(1) + 8.14 = 6.28$$

§2.8 The Derivative as a function.

We have the derivative of f at a

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Let $h = x - a$

then $h + a = x$

$x = a + h$

we get

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This is another definition of derivative.

We can write

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

We see that the derivative is itself a function.

Example Let $f(x) = x^2$.

(a) Find $f'(x)$.

~~$f'(x) = \lim_{h \to 0} f(x)$~~ f

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{(\cancel{x^2} + 2hx + h^2) - \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$= \lim_{h \to 0} (2x + h) = 2x + 0 = 2x.$$

Answer: $f'(x) = 2x$

(b) Find $f'(5)$.

SOLUTION: $f'(5) = 2 \cdot 5 = 10$.

Theorem: If f is differentiable at a (that is, $f'(a)$ exists), then f is continuous at a.

Example: Let $f(x) = |x|$. Let try to find $f'(0)$.

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|0+h| - |0|}{h}$$

$$= \lim_{h \to 0} \frac{|h|}{h}$$

we have

$$|h| = \begin{cases} h & \text{if } h > 0 \\ -h & \text{if } h < 0 \end{cases}$$

- $\displaystyle\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0} 1 = 1$

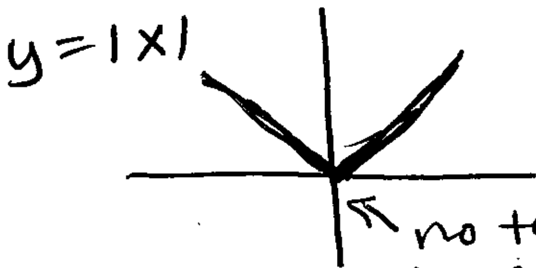
  h is pos so $|h| = h$

- $\displaystyle\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} -1 = -1$

  if $h < 0$, $|h| = -h$

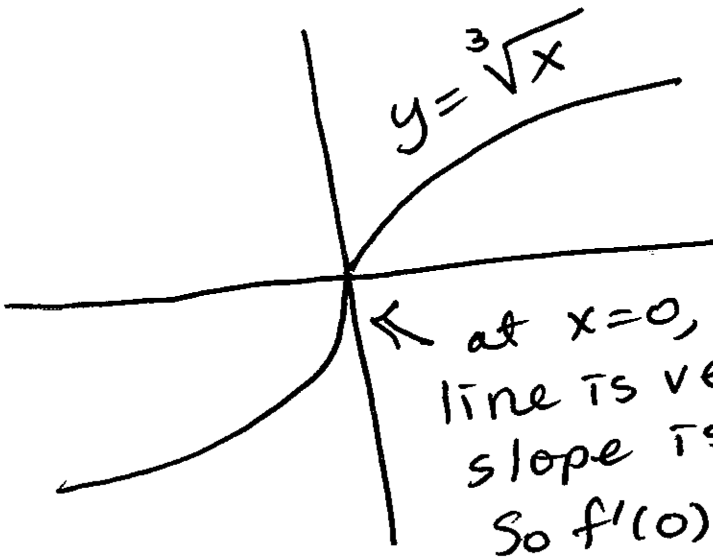
Therefore the limit DNE because the left left and right hand limits do not agree.

Gallery of Continuous functions that are not differentiable at a point.



no tangent line at $x=0$.
$f'(0)$ DNE

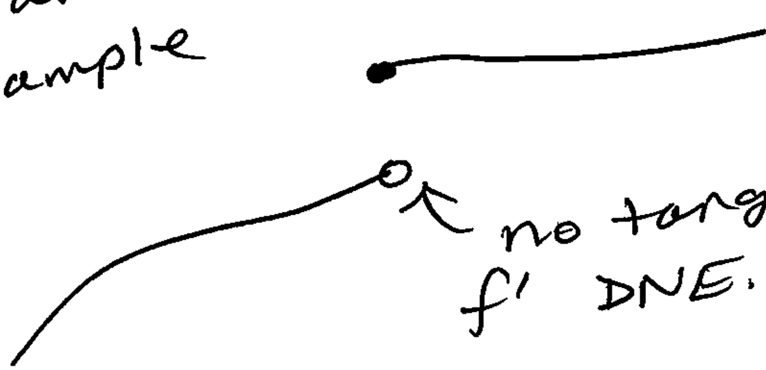
- If there is a sharp edge, the derivative does not exist.



at $x=0$, the tangent line is vertical, so the slope is undefined.
So $f'(0)$ DNE.

If a function is not continuous at a number a, then $f'(a)$ DNE

Example



no tangent line here, f' DNE.

Notation

We can write the derivative of f

$$f'(x), \ \frac{dy}{dx}, \ \frac{df}{dx}, \ Df$$

No homework for § 2.8

Chapter 3 Differentiation Rules.

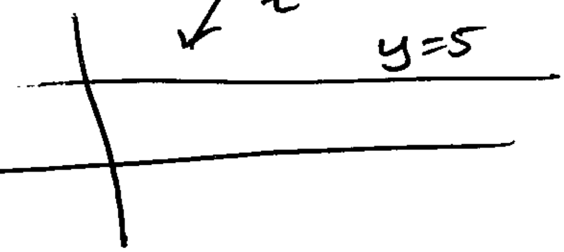
§3.1 Derivatives of Polynomials and Exponential Functions.

Rules of Derivatives

① The Derivative of a constant is zero.

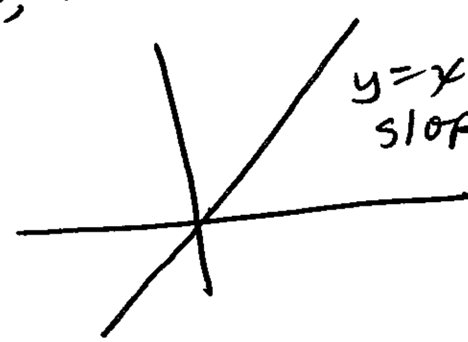
Example: $\frac{d}{dx}(5) = 0.$

slope of tangent line is 0.



② $\frac{d}{dx}(x) = 1$

This means, if $f(x) = x$, then $f'(x) = 1$.



slope = derivative

slope = 1.

③ The Power Rule

$$\frac{d}{dx}(x^n) = n\,x^{n-1}$$

Proof: $x^2-y^2=(x-y)(x+y)$

$$x^3-y^3=(x-y)(x^2+xy+y^2)$$

$$x^4-y^4=(x-y)(x^3+x^2y+xy^2+y^3)$$

$$x^5-y^5=(x-y)(x^4+x^3y+x^2y^2+xy^3+y$$

$$x^n-y^n=(x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2 + \cdots xy^{n-2}+y^{n-1})$$

If $f(x)=x^n$, then

~~$f'(x)=\lim_{h\to}$~~

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

$$= \cancel{\lim_{x\to a}\frac{x^n-x^a}{x-a}} = \lim_{x\to a}\frac{x^n-a^n}{x-a}$$

$$=\lim_{x\to a}\frac{(\cancel{x-a})(x^{n-1}+x^{n-2}a+\cdots xa^{n-2}+a^{n-1})}{(\cancel{x-a})}$$

Put in $x=a$

$$= a^{n-1}+a^{n-2}\cdot a+\cdots a\cdot a^{n-2}+a^{n-1}$$

$$= a^{n-1}+a^{n-1}+\cdots a^{n-1}+a^{n-1}$$

$$= na^{n-1}$$

Example: Find y'.

① $y = x^{10}$

$y' = 10x^9$

② $y = x^{100}$

$y' = 100x^{99}$

③ $y = \sqrt{x}$

$y = x^{1/2}$

$y' = \frac{1}{2}x^{-1/2}$

$y' = \frac{1}{2x^{1/2}}$

$y' = \frac{1}{2\sqrt{x}}$

④ $y = \frac{1}{x^3}$

$y = x^{-3}$

$y' = -3x^{-4}$

$y' = \frac{-3}{x^4}$

More Rules of Differentiation.

④ $\frac{d}{dx}\left(c f(x)\right) = c \frac{d}{dx} f(x)$

c is a constant

Example: $y = 7x^3 + 5$

Find y'

SOLUTION: $y' = 7(3x^2) + 0$

$y' = 21x^2$

⑤ $\frac{d}{dx}\left[f(x) + g(x)\right]$

$= f'(x) + g'(x)$

Example: $y = x^3 + x^4$. Find y'

SOLUTION $y' = 3x^2 + 4x^3$

(6) $\frac{d}{dx}[f(x) - g(x)]$

$$= f'(x) - g'(x).$$

Example Find y'

(a) $y = x^5 - 3x^4 + 7x + 2$

$y' = 5x^4 - 3 \cdot 4x^3 + 7 + 0$

$y' = 5x^4 - 12x^3 + 7$

(b) $y = x^2 - 7x + 2$

$y' = 2x - 7$

(c) $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

$y = x^{1/2} + \frac{1}{x^{1/2}}$

$y = x^{1/2} + x^{-1/2}$

$y' = \frac{1}{2}x^{-1/2} + -\frac{1}{2}x^{-3/2}$

$y' = \frac{1}{2x^{1/2}} + -\frac{1}{2x^{3/2}}$

$$\boxed{y' = \frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}}$$

Aside

$x^{3/2} = \sqrt{x^3} = \sqrt{x \cdot x^2}$

$= x\sqrt{x}$

~~$x^{3/2} = x\sqrt{x}$~~