§3.1 Continued

2/23/2010

Derivatives of Exponential Functions

FORMULA: $\frac{d}{dx} e^x = e^x$

More generally $\frac{d}{dx} 2^x = 2^x \ln 2$

$$\frac{d}{dx} a^x = a^x \ln a$$

Explanation:

Let $f(x) = a^x$

then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{a^{(x+h)} - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x a^h - a^x}{h}$$

$$= \lim_{h \to 0} a^x \left(\frac{a^h - 1}{h} \right)$$

$$= a^x \lim_{h \to 0} \left(\frac{a^h - 1}{h} \right)$$

Aside: If $f(x) = a^x$, what is $f'(0)$.

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

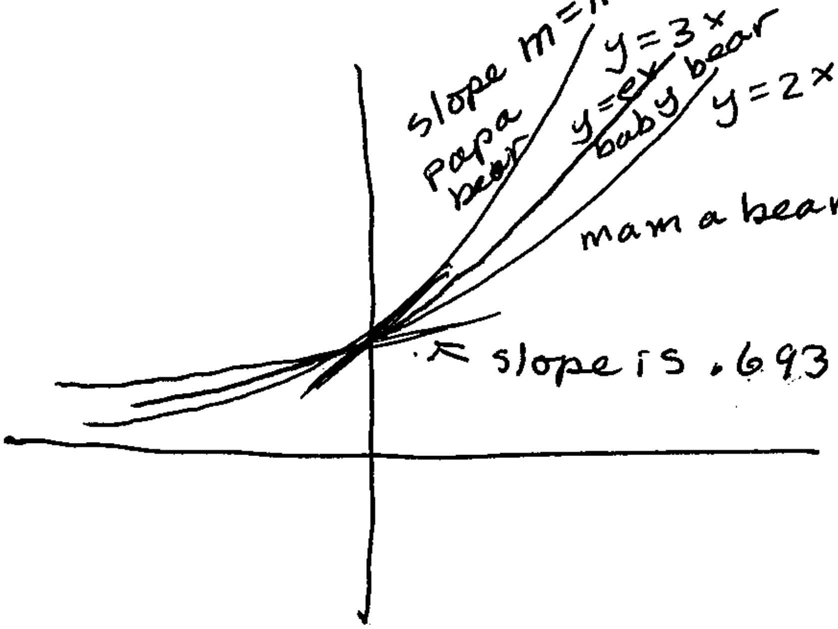
$$= \lim_{h \to 0} \frac{e^{0+h} - e^{0}}{h}$$

$$= \lim_{h \to 0} \frac{e^{h} - 1}{h}$$

Therefore: $f'(x) = a^x f'(0)$.

$f'(0)$ is the slope of the tangent line at $x=0$.

The graph of $y = a^x$



What are the slopes of the tangent lines through $(0,1)$ for $y=a^x$ when a takes different values?

a	a^x	$f'(0)$
2	2^x	0.693
2.1	$(2.1)^x$	0.742
2.2	$(2.2)^x$	.788
2.3	$(2.3)^x$	.833
2.4	$(2.4)^x$	.875
2.5	$(2.5)^x$	.916
2.6	$(2.6)^x$	.956
2.7	$(2.7)^x$	.993
e	e^x	1
2.8	$(2.8)^x$	1.03
2.9	$(2.9)^x$	1.06
3	3^x	1.10

somewhere in between here, $f'(0)=1$.

Define e as the number such that if $f(x)=e^x$, then $f'(0)=1$. $e \approx 2.71828183$

So if $f(x)=e^x$, then $f'(x)=e^x f'(0)$
$= e^x$.

So $\frac{d}{dx} e^x = e^x$.

Aside

We will see later $\frac{d}{dx}(e^{2x}) = 2e^{2x}$

Example Find the equation of the tangent ~~to~~ line to $y = 3x + e^x$ at the point $(0, 1)$.

SOLUTION

To find the slope, take the derivative.

$$f(x) = 3x + e^x$$
$$f'(x) = 3(1) + e^x$$
$$m = f'(0) = 3 + e^0 = 3 + 1 = 4$$

put in $x = 0$.

$$m = 4, \quad x_1 = 0, \quad y_1 = 1$$
$$y - y_1 = m(x - x_1)$$
$$y - 1 = 4(x - 0)$$
$$y - 1 = 4x$$
$$y = 4x + 1.$$

§3.2 ~~The Pr~~

The Product and Quotient Rules

HW §3.2 #1-34, 41-50

First we give some formulas that will proved in §3.3.

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

The Product Rule

$$\frac{d}{dx}[f(x)g(x)] = f'(x)\,g(x) + f(x)\,g'(x)$$

OR

$$= f(x)\,g'(x) + f'(x)\,g(x)$$

Example: Find the derivative.

① $y = x^2 \cdot e^x$ use product rule

$y' = (x^2)'(e^x) + (x^2)(e^x)'$

$y' = 2x e^x + x^2 e^x$

(2) $y = x^3 \sin x$

$y' = (x^3)'(\sin x) + (x^3)(\sin x)'$

$y' = 3x^2 \sin x + x^3 \cos x$

The Quotient Rule

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{[g]^2}$$

Example: Find the derivative.

(a) $y = \dfrac{\sin x}{x^3}$

$$y' = \frac{(\sin x)'(x^3) - (\sin x)(x^3)'}{[x^3]^2}$$

$$y' = \frac{(\cos x)\, x^3 - (\sin x)(3x^2)}{x^6}$$

$$= \frac{x^2(x \cos x - 3 \sin x)}{x^6}$$

$$= \frac{x \cos x - 3 \sin x}{x^4}$$

Example: Find the derivative of $\tan x = \dfrac{\sin x}{\cos x}$

$$\frac{d}{dx}\tan x = \frac{(\sin x)'(\cos x) - (\sin x)(\cos x)'}{(\cos x)^2}$$

$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2}$$

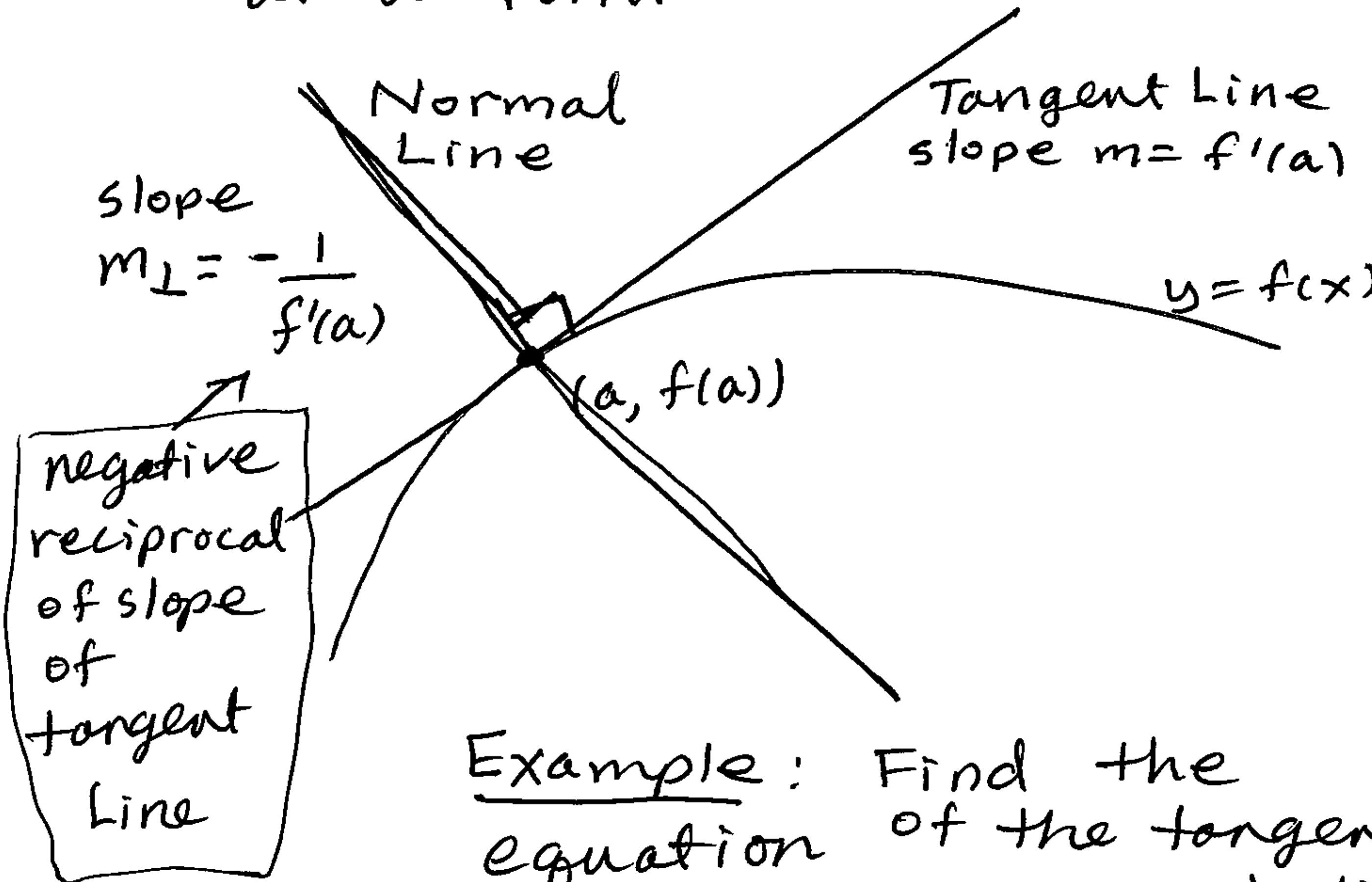
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

$$\boxed{\frac{d}{dx}\tan x = \sec^2 x}$$

The Normal Line to a Curve at a Point



<u>Example</u>: Find the equation of the tangent line and the normal line to $f(x) = x^2$ at the point $(2, 4)$.

<u>SOLUTION</u> $\quad m = f'(a)$ <u>Tangent Line</u>

$$f'(x) = 2x$$

$$m = f'(2) = 2(2) = 4$$

$$x_1 = 2, \ y_1 = 4, \ m = 4$$

$$y - y_1 = m(x - x_1)$$

$$y - 4 = 4(x - 2)$$

$$y - 4 = 4x - 8$$

$$y = 4x - 4$$

Normal Line

$m_\perp = -\frac{1}{4}$

$x_1 = 2, \quad y_1 = 4$

$y = mx + b$

$y = -\frac{1}{4}x + b$

To find b, put in $x_1 = 2, y_1 = 4$

$4 = -\frac{1}{4}(2) + b$

$4 = -\frac{1}{2} + b$

$b = 4 + 1/2 = \frac{8}{2} + \frac{1}{2} = \frac{9}{2}$

$y \cancel{= -\frac{1}{4}}$

$y = -\frac{1}{4}x + \frac{9}{2}$