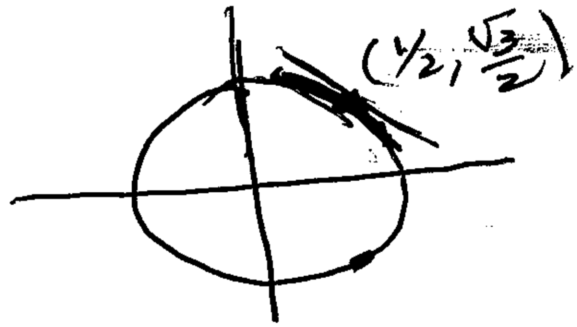
§3.5 Implicit Differentiation

HW §3.5 # 1–30, 33–36,
45–54

$\boxed{\text{Quiz Thurs. } \S\, 3.1\text{–}3.4}$

Example: Let's look at the circle $x^2+y^2=1$



This is not a function. It does not pass the vertical line test.

For a small interval about the point of tangency, y is a function of x. We can write $y=y(x)$.

Let's find $\frac{dy}{dx}$.

$$x^2+(y)^2=1$$

Apply $\frac{d}{dx}$ to both sides

$$\frac{d}{dx}\left(x^2+(y)^2\right)=\frac{d}{dx}(1)$$

chainrule

$$2x+2(y)\cdot\frac{dy}{dx}=0$$

inside function

deriv. inside function.

Solve for $\frac{dy}{dx}$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

FOLLOW-UP

- Find the ~~t~~ equation of the tangent line to $x^2+y^2=1$ at the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

SOLUTION

$$m = \left.\frac{dy}{dx}\right|_{\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)}$$

put in $x = \frac{1}{2}$, $y = \frac{\sqrt{3}}{2}$ to

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{dy}{dx} = -\frac{(1/2)}{(\sqrt{3}/2)} = -\frac{1}{2}\cdot\frac{2}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

$$= -\frac{\sqrt{3}}{3}$$

$x_1 = \frac{1}{2}$, $y_1 = \frac{\sqrt{3}}{2}$, $m = -\frac{\sqrt{3}}{3}$ $\qquad y - y_1 = m(x - x_1)$

$$y - \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{3}\left(x - \frac{1}{2}\right)$$

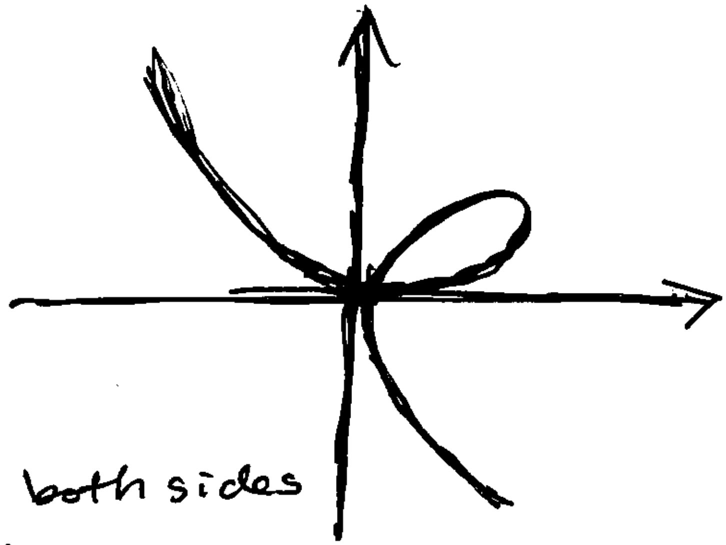
$$y - \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{6}$$

$$y = -\frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{6} + \frac{3\sqrt{3}}{3\cdot 2}$$

$$y = -\frac{\sqrt{3}}{3}x + \frac{4\sqrt{3}}{6} \qquad \boxed{y = -\frac{\sqrt{3}}{3}x + \frac{2\sqrt{3}}{3}}$$

Example The Folium of Descartes

$$x^3+y^3=6xy$$



a) Find $\frac{dy}{dx}$ using implicit differentiation.

Apply d/dx to both sides

$$x^3+y^3 = 6x\cdot y$$

product rule

$$3x^2 + 3y^2\cdot y' = 6\left((x)'(y)+(x)(y)'\right)$$

$$3x^2+3y^2y' = 6(1\cdot y + xy')$$

$$3x^2+3y^2y' = 6y+6xy'$$

solve for y'

$$3y^2y' - 6xy' = 6y-3x^2$$

$$y'(3y^2-6x) = 6y-3x^2$$

$$\boxed{y' = \frac{6y-3x^2}{3y^2-6x}} = \frac{\cancel{3}(2y-x^2)}{\cancel{3}(y^2-2x)}$$

$$\boxed{y' = \frac{2y-x^2}{y^2-2x}}$$

(b) Find the equation of the tangent line at the point (3,3).

SOLUTION To find m, put in $x=3$, $y=3$ to $y' = \frac{2y-x^2}{y^2-2x}$

$$m = \frac{2(3)-(3)^2}{(3)^2-2(3)} = \frac{6-9}{9-6} = \frac{-3}{3} = -1$$

$x_1 = 3,\ y_1 = 3,\ m = 1$

$$y - y_1 = m(x - x_1)$$

$$y - 3 = {}^{-}1 \cdot (x - 3)$$

$$\cancel{y = x - 3 + 3}$$

$$\cancel{y = x}$$

$$y - 3 = -x + 3$$

$$\boxed{y = -x + 6}$$

Explanation:

$$x^3 + y^3 = 6xy$$

if $y = f(x)$, we can write

$$x^3 + (f(x))^3 = 6x f(x)$$

Find $f'(x)$.

$$3x^2 + 3(f(x))^2 f'(x) = 6\overbrace{\Big(\underbrace{x'}_{=1} f(x) + x f'(x)\Big)}^{\text{prod. rule}}$$

<u>etc.</u> This is an alternative notation for the previous problem.

Derivatives of Inverse Trig Functions

You do not need to memorize.

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}(\csc^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$$

Example Find the derivative.

a) $y = \tan^{-1}\sqrt{x}$

$y = \tan^{-1}(x^{1/2})$

$$y' = \frac{1}{1+(x^{1/2})^2} \cdot \left(\frac{1}{2}x^{-1/2}\right)$$

put in $u = x^{1/2}$

$$y' = \left(\frac{1}{1+x}\right)\frac{1}{2\sqrt{x}}$$

This is a composition of functions

$u = x^{1/2}$

$y = \tan^{-1}u$

$$\frac{du}{dx} = \frac{1}{2}x^{-1/2}$$

$$\frac{dy}{du} = \frac{1}{1+u^2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \left(\frac{1}{1+u^2}\right)\left(\frac{1}{2}x^{-1/2}\right)$$

$$= \left(\frac{1}{1+x}\right)\left(\frac{1}{2\sqrt{x}}\right)$$

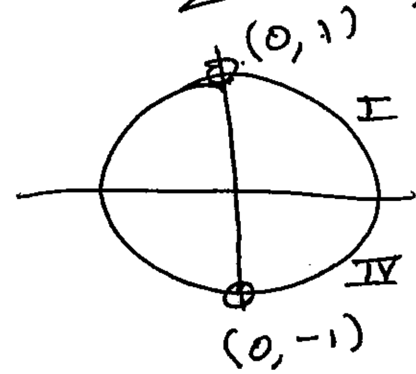
Let's prove $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$.

<u>Proof</u> Let $y = \tan^{-1}x$

by definition $\tan y = x$

Let's apply $\frac{d}{dx}$ to both sides

$$-\frac{\pi}{2} < y < \frac{\pi}{2}$$



$$\frac{d}{dx}\tan y = \frac{d}{dx}x$$

impl. diff.

$$(\sec^2 y)y' = 1$$

$$y' = \frac{1}{\sec^2 y}$$

$$\cancel{y' = \cos^2 y}$$

$$\boxed{y' = \frac{1}{1+x^2}}$$

What is $\sec^2 y$ if $\tan y = x$?

$$\tan^2 y + 1 = \sec^2 y$$

so putting in $\tan y = x$

$$x^2 + 1 = \sec^2 y$$