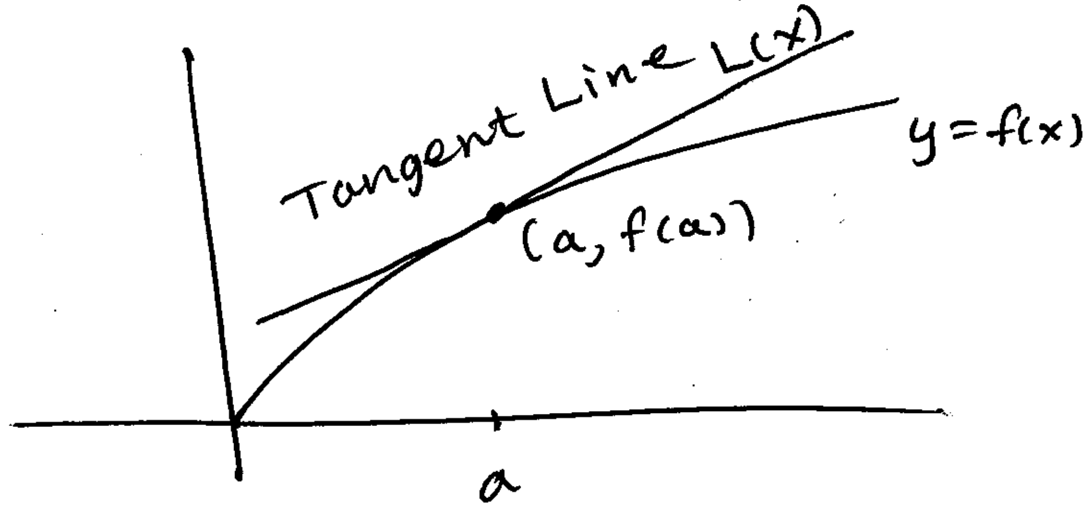
§3.10 Linear Approximations and Differentials

HW §3.10 #1–4, 11–18, 33, 35



When x is close to a, the tangent line, L(x), is a good approximation for f(x).

Let's find a general formula for the tangent line to f at a.

$$y - y_1 = m(x - x_1)$$

$$x_1 = a$$

$$y_1 = f(a)$$

$$m = f'(a)$$

$$y - f(a) = f'(a)(x-a)$$

$$y = f(a) + f'(a)(x-a)$$

We call this the linear approximation to f at a.

$$L(x) = f(a) + f'(a)(x-a)$$

<u>Example</u>: Let $f(x)=\sqrt{x}$.

a) Find the the linear approximation to $f(x)=\sqrt{x}$ at $x=4$.

<u>SOLUTION</u>: $a=4$

$$f(a) = f(4) = \sqrt{4} = 2$$

$$f(x) = x^{1/2}$$

$$f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$f'(a) = f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$$L(x) = f(a) + f'(a)(x-a)$$

$$L(x) = 2 + \frac{1}{4}(x-4)$$

b) Find an approximation for $\sqrt{(4.1)}$ using $L(x)$.

<u>SOLUTION</u> $\sqrt{4.1} = f(4.1) \approx L(4.1)$

$$L(4.1) = 2 + \frac{1}{4}(4.1-4)$$

$$= 2 + (.25)(.1)$$

$$= 2 + .025 = 2.025$$

Using a calculator

$$f(4.1) = \sqrt{4.1} \approx 2.0248$$

The difference is .0002.

Differentials.

Let's look at the formula for $L(x)$.

$$L(x) = f(a) + \underbrace{f'(a)(\overbrace{x-a}^{dx})}_{\text{this is called the differential in } y\text{, denoted } dy}$$

For the last problem, $f(x) = \sqrt{x}$, $a = 4$, we had

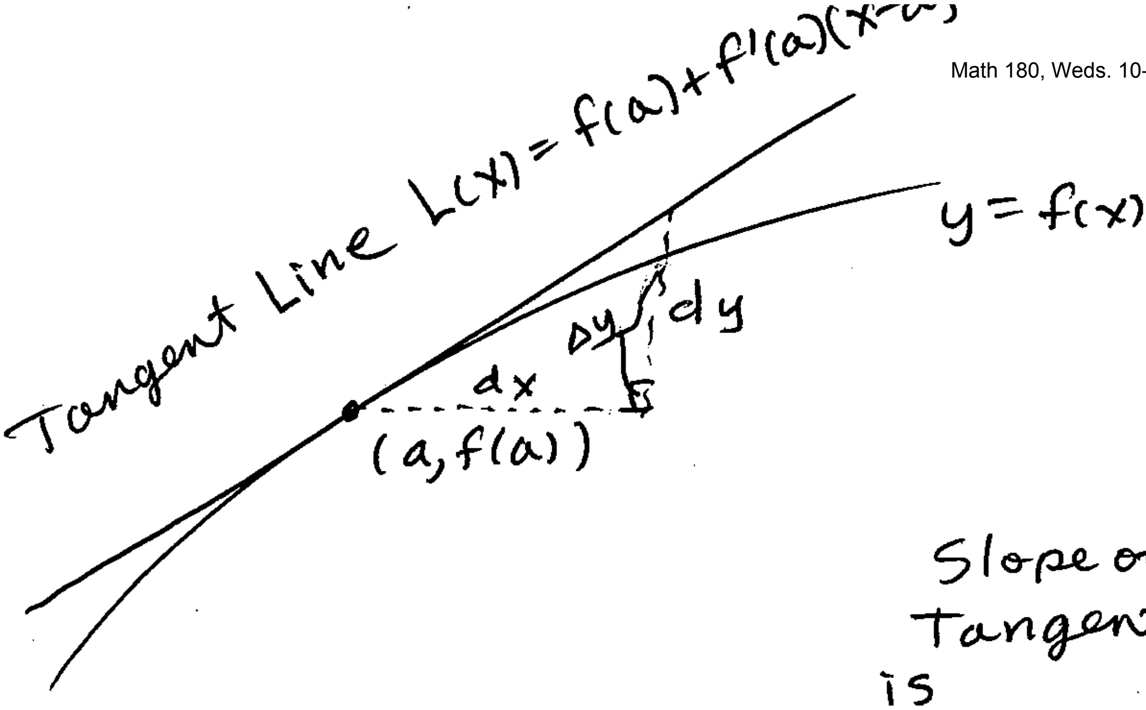
$$L(x) = \underbrace{2}_{\sqrt{4}} + \underbrace{\tfrac{1}{4}(x-4)}_{dy = \frac{1}{4}(x-4)}$$

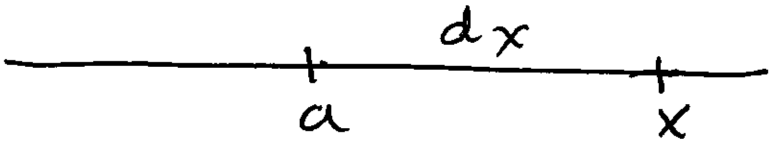
$$L(4.1) = \underbrace{2}_{\sqrt{4}} + \underbrace{\tfrac{1}{4}(4.1-4)}_{dy}$$

We call $x-a$ the differential in x, $dx = x-a$.

So $dy = f'(a)\,dx$

$$dy = \left(\frac{dy}{dx}\right) dx$$





Slope of Tangent Line is

$$\frac{dy}{dx} = \frac{\text{rise}}{\text{run}}$$

run $= dx$

~~rise~~

$$\text{rise} = \left(\frac{dy}{dx}\right)\text{run}$$

$$= \left(\frac{dy}{dx}\right)dx$$

So $dy = $ rise

We have

$$dy = \left(\frac{dy}{dx}\right)dx$$

Define $\Delta y = f(x) - f(a)$

We see that

$\Delta y \approx dy$

when dx is small,

i.e. x is close to a.

Example The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in the calculated value of the volume?

SOLUTION $V = \frac{4}{3}\pi r^3$

$r = 21$ cm

$dr = .05$ cm

Find dV.

$$dV = \left(\frac{dV}{dr}\right) dr$$

Find $\frac{dV}{dr} = \frac{4}{\cancel{3}}\pi \cdot \cancel{3} r^2 = 4\pi r^2$

$\frac{dV}{dr}$ when $r = 21$ is $\frac{dV}{dr} = 4\pi(21)^2$

So $dV = \left(\frac{dV}{dr}\right) dr = 4\pi(21)^2 \cdot (.05)$

$\approx 277 \text{ cm}^3$

b) What is the relative error in volume?

$$\text{relative error} = \frac{dV}{V} = \frac{277}{\left(\frac{4}{3}\pi(21)^3\right)} \approx .007$$

$$V(21) = \frac{4}{3}\pi(21)^3$$

c) What is the relative error in the radius?

$$= \frac{dr}{r} = \frac{.05}{21} \approx .0024$$

d) Let's find a formula relating relative error in volume to relative error in radius.

rel. error in vol $\dfrac{dV}{V} = \dfrac{\left(\frac{dV}{dr}\right)dr}{V} = \dfrac{(4\pi r^2)\,dr}{\left(\frac{4}{3}\pi r^3\right)} = 3\left(\dfrac{dr}{r}\right)$

§3.5 #19

$$e^y \cdot \cos x = 1 + \sin(xy)$$

Find $\frac{dy}{dx}$ using impl. diff.

product rule   chain rule

$$(e^y)'(\cos x) + (e^y)(\cos x)' = 0 + \cos(xy) \cdot (xy)'$$

$$(e^y)(y')(\cos x) + (e^y)(-\sin x) = \cos(xy) \cdot \left[(x)'(y) + (x)(y)'\right]$$

$$e^y \cos x \, y' - e^y \sin x = \cos(xy)(1 \cdot y + x y')$$

$$e^y \cos x \, y' - e^y \sin x = y \cos(xy) + x \cos(xy) \, y'$$

$$e^y \cos x \, y' - x\cos(xy) \, y' = y \cos(xy) + e^y \sin x$$

$$y' \left(e^y \cos x - x \cos(xy)\right) = y \cos(xy) + e^y \sin x$$

$$y' = \frac{y \cos(xy) + e^y \sin x}{e^y \cos x - x \cos(xy)}$$

§3.5 #33 Find y'' by implicit differentiation.

$$9x^2 + y^2 = 9$$

See §3.5 Example 4 p. 211

SOLUTION

$$18x + 2yy' = 0$$

$$2yy' = -18x$$

$$y' = \frac{-18x}{2y}$$

$$y' = -\frac{9x}{y}$$

$$y'' = \frac{(-9x)'(y) - (-9x)(y)'}{(y)^2}$$

$$y'' = \frac{-9y + 9xy'}{y^2}$$

put in $y' = -\frac{9x}{y}$

$$y'' = \left(\frac{-9y + 9x\left(-\frac{9x}{y}\right)}{y^2}\right) \cdot \frac{y}{y}$$

$$y'' = -\frac{9y^2 - 81x^2}{y^3}$$

Note $9x^2+y^2=9$

$$y'' = -\frac{9(y^2 + 9x^2)}{y^3} = -\frac{9(9)}{y^3}$$

$$y'' = -\frac{81}{y^3}$$

§3.5 #17 Find $\frac{dy}{dx}$ using implicit differention.

$$\sqrt{xy} = 1 + x^2 y$$

$$(xy)^{1/2} = 1 + x^2 y$$

$$\frac{1}{2}(xy)^{-1/2} \cdot \Big(\underbrace{(x)'(y) + (x)(y)'}_{\substack{\frac{d}{dx}(xy) \\ \text{use product rule}}}\Big) = 0 + \big((x^2)'(y) + (x^2)(y\big.$$

$$\frac{1}{2}(xy)^{-1/2}(1 \cdot y + xy') = 2xy + x^2 y'$$

$$\frac{y + xy'}{2(xy)^{1/2}} = 2xy + x^2 y'$$

$$y + xy' = (2xy + x^2 y')\, 2(xy)^{1/2}$$

$$y + xy' = 4xy(xy)^{1/2} + 2(xy)^{1/2} x^2 y'$$

$$xy' - 2(xy)^{1/2} x^2 y' = 4xy(xy)^{1/2} - y$$

$$y'(x - 2x^2\sqrt{xy}) = 4xy\sqrt{xy} - y$$

$$y' = \frac{4xy\sqrt{xy} - y}{x - 2x^2\sqrt{xy}}$$

QUIZ Thurs.
§ 3.5, 3.6

§ 3.5 #45

Find the derivative.

$$y = \tan^{-1}\sqrt{x}$$

$$y = \tan^{-1}(x^{1/2})$$

FORMULA

$$\boxed{\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}}$$

$$y' = \left(\frac{1}{1+(x^{1/2})^2}\right) \cdot \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}(x^{1/2})$$

$$y = \tan^{-1}u$$

$$\frac{dy}{du} = \frac{1}{1+u^2}$$

$$u = x^{1/2}$$

$$\frac{du}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$y' = \left(\frac{1}{1+x}\right)\left(\frac{1}{2\sqrt{x}}\right)$$

Math 180 10 March 2010

§ 3.4 #17

$$g(x) = (1+4x)^5 (3+x-x^2)^8$$

product rule

$$g'(x) = \left[(1+4x)^5\right]'(3+x-x^2)^8 + (1+4x)^5\left[(3+x-x^2)^8\right]'$$

$$= 5(1+4x)^4 \cdot 4(3+x-x^2)^8 + (1+4x)^5 \cdot 8(3+x-x^2)^7(1-2x)$$

$$= 4(1+4x)^4(3+x-x^2)^7\left(5(3+x-x^2) + (1+4x)\cdot 2\cdot(1-2x)\right)$$

$$= 4(1+4x)^4(3+x-x^2)^7\left(15+5x-5x^2+2(1-2x+4x-8x^2\right.$$

$$= 4(1+4x)^4(3+x-x^2)^7\left(15+5x-5x^2+2+4x-16x^2\right)$$

$$= 4(1+4x)^4(3+x-x^2)^2\left(-21x^2+9x+17\right)$$

§3.6 #21 $y = 2x \log_{10} \sqrt{x}$

Find y'

$$y = 2x \log_{10} x^{1/2}$$

$$y = 2x\left(\tfrac{1}{2}\right) \log_{10} x$$

$$y = x \log_{10} x$$

Aside

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

$$y' = (x)'(\log_{10} x) + (x)(\log_{10} x)'$$

$$y' = 1 \cdot \log_{10} x + x \cdot \frac{1}{x \ln 10}$$

$$y' = \log_{10} x + \frac{1}{\ln 10}$$

§3.6 #33

Find the equation of the tangent line to $y = \ln(xe^{x^2})$ at the point $(1,1)$.

SOLUTION FIND THE SLOPE.

Find y'. First expand.

$$y = \ln x + \ln e^{x^2}$$

$$y = \ln(x) + x^2$$

$$y' = \frac{1}{x} + 2x$$

$$m = y'(1) = \frac{1}{(1)} + 2(1) = 3$$

Find the equation of the line.

$$y - y_1 = m(x - x_1)$$

$$x_1 = 1, \quad y_1 = 1, \quad m = 3$$

$$y - 1 = 3(x - 1)$$

$$y - 1 = 3x - 3$$

$$\boxed{y = 3x - 2}$$

§3.6 #34 $y = \ln(x^2 - 7)$

Find the tangent line at the point $(2, 0)$.

SOLUTION

Find the slope.

$$y' = \left(\frac{1}{x^2-7}\right)(2x)$$

$$m = y'(2) = \left(\frac{1}{(2)^2-7}\right)2(2)$$

$$= \left(\frac{1}{-3}\right)4 = \frac{-4}{3}$$

$x_1 = 2,\ y_1 = 0,\ m = \frac{-4}{3}$

$$y - y_1 = m(x - x_1)$$

$$y - 0 = \frac{-4}{3}(x - 2)$$

$$y = \frac{-4}{3}x + \frac{8}{3}$$

§3.6 #17 $\quad y = \ln|2 - x - 5x^2|$

SOLUTION

$$y' = \left(\frac{1}{2-x-5x^2}\right)(-1-10x)$$

$$y' = \frac{-1-10x}{2-x-5x^2}$$

Aside

$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$

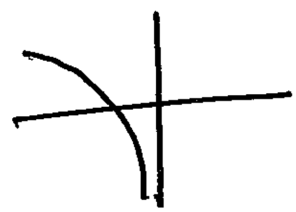
Pf $\quad |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

$$\ln|x| = \begin{cases} \ln x & \text{for } > 0 \\ \ln(-x) & \text{for } x < 0 \\ \text{undef} & x = 0 \end{cases}$$

$$\frac{d}{dx}\ln|x| = \begin{cases} 1/x & \text{for } x > 0 \\ \frac{1}{(-x)}\cdot -1 = \frac{1}{x} & \text{for } x < 0 \\ \text{undef} & x = 0 \end{cases}$$

$y = \ln x$

$y = \ln(-x)$



$y = \ln|x|$

§3.6 #5 Differentiate.

$$y = \log_2(1-3x)$$

$$y' = \frac{1}{(1-3x)\ln 2} \cdot (-3)$$

$\frac{d}{dx}(1-3x)$

Aside

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

$$y' = \frac{-3}{(1-3x)\ln 2}$$

Quiz Thurs.
§ 3.5, 3.6

§ 3.5 #15 Find $\frac{dy}{dx}$ using implicit differentiation.

$$e^{(x/y)} = x - y$$

SOLUTION

$$\underbrace{e^{(x/y)} \cdot \left(\frac{x}{y}\right)'}_{\text{chain rule}} = 1 - y'$$

$$e^{x/y}\Bigg(\underbrace{\frac{x'y - xy'}{y^2}}_{\text{Quotient Rule}}\Bigg) = 1 - y'$$

$$e^{x/y}\left(\frac{1 \cdot y - xy'}{y^2}\right) = 1 - y'$$

$$e^{x/y}(y - xy') = (1 - y')y^2$$

Solve for y'

$$e^{x/y}(y - xy') = y^2 - y^2 y'$$

$$ye^{x/y} - xy'e^{x/y} = y^2 - y^2 y'$$

$$y^2 y' - xy'e^{x/y} = y^2 - ye^{x/y}$$

$$y'(y^2 - xe^{x/y}) = y^2 - ye^{x/y}$$

$$y' = \frac{y^2 - ye^{x/y}}{y^2 - xe^{x/y}}$$