Integration of Rational Functions

In class we studied the integrals of rational functions where the denominator is a quadratic, $ax^2 + bx + c$. If the numerator has degree greater or equal to the degree of the denominator, we first do long division. Then we can break up this type of integral into three different cases. An example of each case is given as well as two homework problems for each case.

Case I: The quadratic in the denominator has two distinct roots ($b^2 - 4ac > 0$)

EXAMPLE: $\displaystyle\int \frac{3x+7}{x^2+6x+5}\,dx$

$$b^2 - 4ac = (6)^2 - 4(1)(5) = 16 > 0$$

Partial Fraction Decomposition:

$$\frac{3x+7}{x^2+6x+5} = \frac{3x+7}{(x+1)(x+5)} = \frac{A}{x+1} + \frac{B}{x+5}$$

Clear the fraction.

$$3x + 7 = A(x+5) + B(x+1)$$

Let $x = -5$

$$3(-5) + 7 = A(-5+5) + B(-5+1)$$

$$-8 = -4B, \quad B = 2$$

Let $x = -1$

$$3(-1) + 7 = A(-1+5) + B(-1+1)$$

$$4 = 4A, \quad A = 1$$

$$\frac{3x+7}{(x+1)(x+5)} = \frac{1}{x+1} + \frac{2}{x+5}$$

$$\int \frac{3x+7}{(x+1)(x+5)}\,dx = \int \left(\frac{1}{x+1} + 2\cdot\frac{1}{x+5}\right)\,dx$$
$$= \ln|x+1| + 2\ln|x+5| + C \quad \square$$

HOMEWORK:

1. $\displaystyle\int \frac{5x-9}{x^2+6x+8}\,dx$
2. $\displaystyle\int \frac{x+9}{x^2-x-6}\,dx$

Case II: The quadratic in the denominator has one root $(b^2-4ac=0)$

EXAMPLE: $\displaystyle\int \frac{3x+7}{x^2+6x+9}\,dx$

$$b^2-4ac = (6)^2 - 4(1)(9) = 0$$

Partial Fraction Decomposition:

$$\frac{3x+7}{x^2+6x+9} = \frac{3x+7}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2}$$

Clear the fraction.

$$3x+7 = A(x+3) + B$$

Let $x=-3$

$$3(-3)+7 = B, \quad B = -2$$

Let $x=0$

$$3(0)+7 = A(0+3) + B$$
$$7 = 3A-2, \quad 3A = 9, \quad A = 3$$

$$\frac{3x+7}{(x+3)^2} = \frac{3}{x+3} - \frac{2}{(x+3)^2}$$

$$\begin{aligned}
\int \frac{3x+7}{(x+3)^2}\,dx &= \int \left(\frac{3}{x+3} - 2(x+3)^{-2}\right)\,dx \\
u &= x+3 \\
du &= dx \\
&= \int \left(\frac{3}{u} - 2(u)^{-2}\right)\,du \\
&= 3\ln|u| - 2\cdot\frac{(u)^{-1}}{-1} + C \\
&= 3\ln|x+1| + \frac{2}{(x+3)} + C \quad \square
\end{aligned}$$

HOMEWORK:

3. $\displaystyle\int \frac{x+5}{x^2+4x+4}\,dx$

4. $\displaystyle\int \frac{3x-2}{x^2+10x+25}\,dx$

Case III: The quadratic in the denominator has no real roots ($b^2 - 4ac < 0$)

EXAMPLE: $\displaystyle\int \frac{3x+7}{x^2+2x+5}\,dx$

$$b^2 - 4ac = (2)^2 - 4(1)(5) = -16 < 0$$

$$\int \frac{3x+7}{x^2+2x+5}\,dx$$

$$\begin{aligned}
&\text{Complete the square}\\
&= \int \frac{3x+7}{(x^2+2x+1)-1+5}\, dx\\
&= \int \frac{3x+7}{(x+1)^2+4}\, dx
\end{aligned}$$

$$\begin{aligned}
u &= x+1\\
dx &= du\\
x &= u-1
\end{aligned}$$

$$\begin{aligned}
&= \int \frac{3(u-1)+7}{(u)^2+4}\, du\\
&= \int \frac{3u+4}{u^2+4}\, du\\
&= \int \frac{3u}{u^2+4}\, du + \int \frac{4}{u^2+4}\, du
\end{aligned}$$

$$\int \frac{3u}{u^2+4}$$

$$\begin{aligned}
\xi &= u^2+4\\
d\xi &= 2u\ du\\
\frac{1}{2}d\xi &= u\ du
\end{aligned}$$

$$\begin{aligned}
&= 3\cdot\frac{1}{2}\int \frac{1}{\xi}\, d\xi\\
&= \frac{3}{2}\ln|\xi| + C\\
&= \frac{3}{2}\ln|u^2+4| + C\\
&= \frac{3}{2}\ln|(x+1)^2+4| + C
\end{aligned}$$

Formula: $\int \frac{1}{u^2+a^2}\, du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$

$$\begin{aligned}\int \frac{4}{u^2+4}\,du \\ &= 4\cdot\frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) \\ &= 2\tan^{-1}\left(\frac{x+1}{2}\right)\end{aligned}$$

$$\int \frac{3x+7}{x^2+2x+5}\,dx = \frac{3}{2}\ln|(x+1)^2+4| + 2\tan^{-1}\left(\frac{x+1}{2}\right) + C \quad \square$$

HOMEWORK:

5. $\displaystyle\int \frac{x+9}{x^2+4x+13}\,dx$
6. $\displaystyle\int \frac{2x-1}{x^2+6x+13}\,dx$