§7.4 Write out the partial fraction decomposition. Do not determine the values of the coefficients.

1(a) $$\frac{2x}{(x+3)(3x+1)} = \frac{A}{x+3} + \frac{B}{3x+1}$$

(b) $$\frac{1}{x^3+2x^2+x} = \frac{1}{x(x^2+2x+1)} = \frac{1}{x(x+1)^2}$$

$$= \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

3a) $$\frac{x^4+1}{x^5+4x^3} = \frac{x^4+1}{x^3(x^2+4)} =$$

irreducible

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+4}$$

b) $$\frac{1}{(x^2-9)^2} = \frac{1}{((x-3)(x+3))^2} = \frac{1}{(x-3)^2(x+3)^2}$$

$$= \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{x+3} + \frac{D}{(x+3)^2}$$

5a) $$\frac{x^4}{x^4-1} = 1 + \frac{\cancel{x^4}\ 1}{(x^2-1)(x^2+1)} = 1 + \frac{\cancel{x^4}\ 1}{(x-1)(x+1)(x^2+1)}$$

irred.

answer:

$$= \boxed{1 + \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}}$$

deg. num = deg denom.

$$1 + \frac{1}{x^4-1}$$

$$\begin{array}{r|l} & 1 \\ x^4-1 & x^4+0x^3+0x^2+0x+0 \\ & -(x^4 \qquad\qquad\qquad -1) \\ \hline & \qquad\qquad\qquad\qquad 1 \end{array}$$

Evaluate the integral.

(7) $\int \frac{x}{x-6}\,dx$

deg num = deg denom.

$$\begin{array}{r} 1 + 6/x-6 \\ x-6\,\overline{)\,x+0} \\ -(x-6) \\ \hline 6 \end{array}$$

$$= \int \left(1 + \frac{6}{x-6}\right) dx$$

$$= \boxed{x + 6\ln|x-6| + C}$$

(9) $\int \frac{(x-9)}{(x+5)(x-2)}\,dx$

$$\frac{x-9}{(x+5)(x-2)} = \frac{A}{x+5} + \frac{B}{x-2}$$

$$x-9 = A(x-2) + B(x+5)$$

$\underline{\underline{x=2}}$

$$2-9 = A(0) + B(2+5)$$

$$7B = -7, \quad B = -1$$

$\underline{\underline{x=-5}}$ $\quad -5-9 = A(-5-2) + B(-5+5)$

$$-14 = -7A$$

$$A = 2$$

$$\int \left(\frac{2}{x+5} - \frac{1}{x-2}\right) dx = 2\ln|x+5| - \ln|x-2| + C$$

(11) $\int_2^3 \frac{1}{x^2-1}\,dx$

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$1 = A(x+1) + B(x-1)$

$\underline{\underline{x=1}}$

$1 = A(1+1) + B(1-1)$

$2A = 1, \quad A = 1/2$

$\underline{\underline{x=-1}}$

$1 = A(-1+1) + B(-1-1)$

$1 = -2B, \quad B = -1/2$

$$\int \frac{1}{2}\left(\frac{1}{x-1}\right) + \frac{-1}{2}\left(\frac{1}{x+1}\right)dx$$

$$= \left[\frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1|\right]_2^3$$

$$= \left(\frac{1}{2}\ln(3-1) - \frac{1}{2}\ln(3+1)\right) - \left(\frac{1}{2}\ln(2-1) - \frac{1}{2}\ln(2+1)\right)$$

$$= \frac{1}{2}\ln 2 - \frac{1}{2}\ln 4 - \frac{1}{2}\underbrace{\ln 1}_{=0} + \frac{1}{2}\ln 3$$

$$= \frac{1}{2}\ln 2 - \frac{1}{2}\ln 2^2 + \frac{1}{2}\ln 3$$

$$= \frac{1}{2}\ln 2 - \frac{1}{2}(2)\ln 2 + \frac{1}{2}\ln 3$$

$$= -\frac{1}{2}\ln 2 + \frac{1}{2}\ln 3$$

$$= \frac{1}{2}(\ln 3 - \ln 2) = \frac{1}{2}\ln \frac{3}{2}$$

(13) $\int \frac{ax}{x^2-bx}\,dx$

$$\frac{ax}{x^2-bx} = \frac{a\cancel{x}}{\cancel{x}(x-b)} = \frac{A}{x} + \frac{B}{x-b}$$

we can just cancel the x's and not do all this work!

$$\boxed{\begin{array}{l} ax = A(x-b) + Bx \\ \underline{x=b} \\ ab = A(b-b) + Bb; \quad ab = Bb \\ \qquad\qquad\qquad\qquad\quad B = a \\ \underline{\underline{x=0}} \quad 0 = A(0-b) + B(0) \\ \qquad\qquad A = 0 \end{array}}$$

$$\int \frac{a}{x-B}\,dx = a\ln|x-b| + C$$

(15) $\displaystyle\int_3^4 \frac{x^3-2x^2-4}{x^3-2x^2}\,dx$

deg num = deg denom

$$\frac{x^3-2x^2-4}{x^3-2x^2} = \frac{x^3-2x^2}{x^3-2x^2} - \frac{4}{x^3-2x^2}$$

$$= 1 - \frac{4}{x^3-2x^2}$$

$$\frac{-4}{x^3-2x^2} = \frac{-4}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$$

$-4 = Ax(x-2) + B(x-2) + Cx^2$

$x = 0$

$-4 = -2B,\quad \boxed{B = 2}$

$x = 2$

$-4 = C(2)^2,\quad \boxed{C = -1}$

$\cancel{x=1}\quad x = 1$

$-4 = A(1)(1-2) + B(1-2) + C(1)^2$

$-4 = -A - B + C$

$-4 = -A - 2 + -1$

$-4 = -A - 3$

$-1 = -A,\quad A = 1$

$$\int\left(1 + \frac{1}{x} + \frac{2}{x^2} + \frac{-1}{x-2}\right)dx$$

$$= \int\left(1 + \frac{1}{x} + 2x^{-2} - \frac{1}{x-2}\right)dx$$

$$= \left[x + \ln|x| + 2\frac{x^{-1}}{-1} - \ln|x-2|\right]_3^4$$

$$= \left[x + \ln|x| - \frac{2}{x} - \ln|x-2|\right]_3^4$$

$$= \left[4 + \ln 4 - 2/4 - \ln(4-2)\right] - \left[3 + \ln 3 - \frac{2}{3} - \ln(3-2)\right]$$

$$= (4 + \ln 4 - 1/2 - \ln 2) - (3 + \ln 3 - 2/3 - \overset{=0}{\ln 1})$$

$$= 4 + \ln 2^2 - 1/2 - \ln 2 - 3 - \ln 3 + 2/3$$

$$= 7/6 + \ln 2 - \ln 3 = 7/6 + \ln(2/3)$$

(17) $\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy$

$$\frac{4y^2-7y-12}{y(y+2)(y-3)} = \frac{A}{y} + \frac{B}{y+2} + \frac{C}{y-3}$$

$$4y^2-7y-12 = A(y+2)(y-3) + By(y-3) + Cy(y+2)$$

- $y=0$ $\quad -12 = -6A, \quad \boxed{A=2}$
- $y=-2$ $\quad 4(-2)^2-7(-2)-12 = 10B$
  $10B = 18, \quad B = \frac{18}{10}, \quad \boxed{B = \frac{9}{5}}$
- $y=3$ $\quad 4(3)^2-7(3)-12 = 15C$
  $15C = 3, \quad \boxed{C = 1/5}$

$$= \int_1^2 \left(\frac{2}{y} + \frac{9}{5}\cdot\frac{1}{y+2} + \frac{1}{5}\cdot\frac{1}{y-3}\right)dy$$

$$= \left[2\ln|y| + \frac{9}{5}\ln|y+2| + \frac{1}{5}\ln|y-3|\right]_1^2$$

$$= \left[2\ln 2 + \frac{9}{5}\ln(2+2) + \frac{1}{5}\ln|2-3|\right] - \left[2\ln 1 + \frac{9}{5}\ln|1+2| + \frac{1}{5}\ln|1-3|\right]$$

$$= 2\ln 2 + \frac{9}{5}\ln 4 + \frac{1}{5}\overset{=0}{\ln 1} - 2\overset{=0}{\ln 1} \mp \frac{9}{5}\ln 3 \mp \frac{1}{5}\ln 2$$

$$\boxed{\ln 4 = \ln 2^2 = 2\ln 2}$$

$$= 2\ln 2 + \frac{9}{5}\cdot 2\ln 2 - \frac{9}{5}\ln 3 - \frac{1}{5}\ln 2$$

$$= \frac{10}{5}\ln 2 + \frac{18}{5}\ln 2 - \frac{9}{5}\ln 3 - \frac{1}{5}\ln 2$$

$$= \boxed{\frac{27}{5}\ln 2 - \frac{9}{5}\ln 3}$$

OR

$$= \frac{9}{5}(3\ln 2 - \ln 3) = \frac{9}{5}(\ln 2^3 - \ln 3) = \boxed{\frac{9}{5}\ln\left(\frac{8}{3}\right)}$$

§7.4 #19 $\displaystyle\int \frac{1}{(x+5)^2(x-1)}\,dx$

$$\frac{1}{(x+5)^2(x-1)} = \frac{A}{x+5} + \frac{B}{(x+5)^2} + \frac{C}{x-1}$$

$$1 = A(x+5)(x-1) + B(x-1) + C(x+5)^2$$

- $x=1$ $\quad 1 = C(1+5)^2$

  $36C = 1, \quad \boxed{C = \frac{1}{36}}$

- $x=-5$ $\quad 1 = B(-5-1)$

  $-6B = 1, \quad \boxed{B = -\frac{1}{6}}$

- $x=0$ $\quad 1 = -5A - B + 25C$

  $1 = -5A - \left(-\frac{1}{6}\right) + 25\left(\frac{1}{36}\right)$

  $-5A = 1 - \frac{1}{6} - \frac{25}{36}$

  $-5A = \frac{36}{36} - \frac{6}{36} - \frac{25}{36}$

  $-5A = \frac{5}{36}, \quad \boxed{A = -\frac{1}{36}}$

$$= \int \left[\left(\frac{-1}{36}\right)\cdot\frac{1}{x+5} + \left(\frac{-1}{6}\right)(x+5)^{-2} + \left(\frac{1}{36}\right)\cdot\frac{1}{x-1}\right] dx$$

$$= \frac{-1}{36}\ln|x+5| - \frac{1}{6}\,\frac{(x+5)^{-1}}{-1} + \frac{1}{36}\ln|x-1| + C$$

$$= \boxed{\frac{-1}{36}\ln|x+5| + \frac{1}{6}\cdot\left(\frac{1}{x+5}\right) + \frac{1}{36}\ln|x-1| + C}$$

§7.4 #21 $\int \frac{x^3+4}{x^2+4}\,dx$ deg num $\geq$ deg denom.

$$\begin{array}{r|l} & x + 0 + \frac{-4x+4}{x^2+4} \\ x^2+4 & x^3 + 0\cdot x^2 + 0\cdot x + 4 \\ & -(x^3 + \quad 4x) \downarrow \\ \hline & \quad -4x+4 \end{array}$$

$$= \int \left(x + \frac{-4x+4}{x^2+4} \right) dx$$

$$= \int \left(\overset{\text{III}}{x} + \overset{\text{I}}{\frac{-4x}{x^2+4}} + \overset{\text{II}}{\frac{4}{x^2+4}} \right) dx$$

I $\int \frac{-4x}{x^2+4}\,dx$

$u = x^2+4$

$\cancel{du = 2x\,dx}$ $du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$-4\cdot\frac{1}{2}\int \frac{1}{u}\,du = -2\ln|u|$$
$$= -2\ln|x^2+4| + C_1$$

II $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C_2$

$$4\int \frac{1}{x^2+4}\,dx = 4\left(\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right) + C_2$$
$$= 2\tan^{-1}\left(\frac{x}{2}\right) + C_2$$

III $\int x\,dx = \frac{x^2}{2} + C_3$

Answer.

$$\frac{x^2}{2} - 2\ln|x^2+4| + 2\tan^{-1}\left(\frac{x}{2}\right) + C$$

§7.4 #23 $\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx$

$$\frac{5x^2+3x-2}{x^3+2x^2} = \frac{5x^2+3x-2}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$$

$$5x^2+3x-2 = Ax(x+2) + B(x+2) + Cx^2$$

- $x=0$ $\quad -2 = 2B, \ \boxed{B=-1}$
- $x=-2$ $\quad 5(-2)^2+3(-2)-2 = C(-2)^2$

  $4C = 12, \ \boxed{C=3}$
- $x=1$

  $5(1)^2+3(1)-2 = A(1)(1+2)+B(1+2)+C(1)^2$

  $6 = 2A+3B+C$

  $6 = 2A+3(-1)+(3)$

  $2A = 6, \ \boxed{A=3}$

$$= \int \left(\frac{3}{x} - x^{-2} + \frac{3}{x+2}\right) dx$$

$$= 3\ln|x| - \frac{x^{-1}}{-1} + 3\ln|x+2| + C$$

$$= 3\ln|x| + \frac{1}{x} + 3\ln|x+2| + C$$

§7.4 #25 $\int \frac{10}{(x-1)(x^2+9)}\,dx$

$$\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9}$$

$$10 = A(x^2+9) + (Bx+C)(x-1)$$

- $x=1$ $\quad 10 = 10A, \quad \boxed{A=1}$
- $x=0$ $\quad 10 = 9A + (B\cdot 0 + C)(0-1)$

$$10 = 9(1) + -C$$

$$\boxed{C = {}^{-}1}$$

put in $A=1, C={}^{-}1$

$$10 = x^2+9 + (Bx \mp 1)(x-1)$$

- $x=2$ $\quad 10 = 2^2+9 + (B\cdot 2 \mp 1)(2-1)$

$$10 = 4+9 + \cancel{13}\ (2B\mp 1)(1)$$

$$10 = 13 + 2B \mp 1, \qquad 10 = 2B + 12$$

$$2B = -2$$

$$\boxed{B=-1}$$

$$= \int \left(\frac{1}{x-1} + \frac{-x+1}{x^2+9} \right) dx$$

$$= \int \left(\frac{1}{x-1}^{\text{(I)}} + \frac{-x}{x^2+9}^{\text{(II)}} + \frac{-1}{x^2+9}^{\text{(III)}} \right) dx$$

(I) $\int \frac{1}{x-1}\,dx = \ln|x-1| + C_1$

(II) $-\int \frac{x}{x^2+9}\,dx = -\frac{1}{2}\int \frac{1}{u}\,du = \frac{-1}{2}\ln|u| = \frac{-1}{2}\ln(x^2+9) + C_2$

$u = x^2$
$du = 2x\,dx$
$\frac{1}{2}\,du = x\,dx$

(III) $-\int \frac{1}{x^2+9}\,dx = {}^{-}\int \frac{1}{x^2+3^2}\,dx = {}^{-}\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C_3$

Answer:

$$\ln|x-1| - \frac{1}{2}\ln(x^2+9) + {}^{-}\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

§7.4 #27 $\displaystyle\int \frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)}\,dx$

$$\frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$$

$$x^3+x^2+2x+1 = (Ax+B)(x^2+2) + (Cx+D)(x^2+1)$$

$$x^3+x^2+2x+1 = Ax^3 + 2Ax + Bx^2 + 2B + Cx^3 + Cx + Dx^2 + D$$

$$x^3+x^2+2x+1 = (A+C)x^3 + (B+D)x^2 + (2A+C)x + (2B+D)$$

$A+C=1$

$B+D=1$

$2A+C=2$

$2B+D=1$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 & 2 \\ 0 & 2 & 0 & 1 & 1 \end{array}\right] \xrightarrow[\to R_3]{-2R_1+R_3} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{-2R_2+R_4\to R_4} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{array}\right] \xrightarrow{R_3+R_1\to R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{array}\right]$$

$$\xrightarrow{-R_4+R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{array}\right] \begin{array}{l} -R_4\to R_4 \\ -R_3\to R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$A=1,\ B=0,\ C=0,\ D=1$

$$= \int \left(\overset{\mathrm{I}}{\frac{x}{x^2+1}} + \overset{\mathrm{II}}{\frac{1}{x^2+2}} \right) dx$$

I $\displaystyle\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C_1$

$\qquad = \frac{1}{2}\ln(x^2+1) + C_1$

$u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

II $\displaystyle\int \frac{1}{x^2+(\sqrt{2})^2}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C_2$

Answer: $\displaystyle \frac{1}{2}\ln(x^2+1) + \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$

§7.4 #29 $\displaystyle\int \frac{x+4}{x^2+2x+5}\,dx$

x^2+2x+5 irreducible.

Complete the square:

$$x^2+2x+5 = (x^2+2x+1)-1+\cancel{4}5$$
$$\cancel{= x^2+2x} \quad = (x+1)^2+4$$

$$= \int \frac{x+4}{(x+1)^2+4}\,dx$$

$u = x+1$, $du = dx$, $x = u-1$

$$= \int \frac{(u-1)+4}{u^2+4}\,du = \int \frac{u+\cancel{4}\,3}{u^2+4}\,du$$

$$= \int \left(\frac{u}{u^2+4}\ \mathrm{I} + \frac{3\cancel{4}}{u^2+4}\ \mathrm{II} \right) du$$

I $\displaystyle\int \frac{u}{u^2+4}\,du = \frac{1}{2}\int \frac{1}{w}\,dw = \frac{1}{2}\ln|w| + C,$

$w = u^2+4$

$dw = 2u\,du$

$\frac{1}{2}dw = u\,du$

$$= \frac{1}{2}\ln(u^2+4)+C,$$
$$= \frac{1}{2}\ln\left((x+1)^2+4\right)+C,$$
$$= \frac{1}{2}\ln(x^2+2x+5)+C,$$

II $\displaystyle\int \frac{\cancel{4}\,3}{u^2+2^2}\,du = \cancel{4}\,3\left(\frac{1}{2}\tan^{-1}\frac{u}{2}\right)+C_2$

$$= \frac{3}{2}\tan^{-1}\frac{x+1}{2} + C_2$$

Answer:

$$\frac{1}{2}\ln(x^2+2x+5) + \frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+C$$

§7.4 #31 $\int \frac{1}{x^3-1}\,dx$

$$\frac{1}{x^3-1} = \frac{1}{(x-1)(x^2+x+1)}$$

$$= \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

$$1 = A(x^2+x+1) + (Bx+C)(x-1)$$

- $x=1$ $\quad 1 = 3A, \ \boxed{A = \frac{1}{3}}$
- $x=0$ $\quad 1 = A - C$

$$1 = \frac{1}{3} - C$$

$$2/3 = -C, \ \boxed{C = -2/3}$$

- $x=-1$ $\quad 1 = A((-1)^2+(-1)+1) + (B(-1)+C)(-1-1)$

$$1 = A + 2B - 2C$$

$$1 = \frac{1}{3} + 2B - 2(-2/3)$$

$$1 = 2B + \frac{5}{3}, \quad 2B = -2/3, \ \boxed{B = \frac{-1}{3}}$$

$$= \int \left(\frac{1}{3}\left(\frac{1}{x-1}\right) + \frac{-1/3\,x - 2/3}{x^2+x+1} \right) dx$$

$$= \int \frac{1}{3}\left(\frac{1}{x-1}\right) - \frac{1}{3}\left(\frac{x+2}{(x^2+x+\frac{1}{4}) - \frac{1}{4} + 1}\right) dx$$

$$= \int \frac{1}{3}\left(\frac{1}{x-1}\right) - \frac{1}{3}\left(\frac{x+2}{(x+\frac{1}{2})^2 + \frac{3}{4}}\right) dx \quad *$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{(x+1/2) + 3/2}{(x+1/2)^2 + 3/4}\,dx \quad u = x + 1/2, \ du = dx \quad *$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{u + 3/2}{u^2 + 3/4}\,du$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{3 \cdot 2}\int \frac{2u}{u^2+3/4}\,du - \frac{1}{3}\cdot\frac{3}{2}\int \frac{1}{u^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,du$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln|u^2+3/4| - \frac{1}{2}\,\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}u\right) + C$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+x+1) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right) + C$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+x+1) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

§7.4 #33 $\int_0^1 \frac{x^3+2x}{x^4+4x^2+3}\,dx$

$$\frac{x^3+2x}{x^4+4x^2+3} = \frac{x^3+2x}{(x^2+3)(x^2+1)} = \frac{Ax+B}{x^2+3} + \frac{Cx+D}{x^2+1}$$

$$x^3+2x = (Ax+B)(x^2+1) + (Cx+D)(x^2+3)$$

$$x^3+2x = Ax^3+Ax+Bx^2+B+Cx^3+3Cx+Dx^2+3D$$

$$x^3+2x = (A+C)x^3+(B+D)x^2+(A+3C)x+(B+3D)$$

$A+C=1$

$B+D=0$

$A+3C=2$

$B+3D=0$

$B+D=0$

$-(B+3D)=0$

$-2D=0$

$\boxed{D=0}$

$B+D=0$

$B+0=0$

$\boxed{B=0}$

$A+C=1$

$-(A+3C)=2)$

$-2C=-1$

$\boxed{C=1/2}$

$A+C=1$

$A+1/2=1$

$\boxed{A=1/2}$

$$= \int_0^1 \left(\frac{1}{2}\frac{x}{x^2+3} + \frac{1}{2}\frac{x}{x^2+1}\right)dx$$

$$= \int_0^1 \left(\frac{1}{2}\cdot\frac{1}{2}\frac{2x}{x^2+3} + \frac{1}{2}\cdot\frac{1}{2}\frac{2x}{x^2+1}\right)dx$$

I $u=x^2+3$, $du=2x\,dx$

II $u=x^2+1$, $du=2x\,dx$

$$= \frac{1}{4}\ln(x^2+3) + \frac{1}{4}\ln(x^2+1)\Big|_0^1$$

$$= \frac{1}{4}\ln(1^2+3) + \frac{1}{4}\ln(1^2+1) - \left(\frac{1}{4}\ln(0^2+3) + \frac{1}{4}\ln(0^2+1)\right)$$

$$= \frac{1}{4}\ln 4 + \frac{1}{4}\ln 2 - \frac{1}{4}\ln 3 + -\frac{1}{4}\overbrace{\ln 1}^{=0}$$

$$= \frac{1}{4}(\ln 2^2 + \ln 2 - \ln 3) = \frac{1}{4}(2\ln 2 + \ln 2 - \ln 3)$$

$$= \frac{1}{4}(3\ln 2 - \ln 3) = \frac{1}{4}(\ln 2^3 - \ln 3)$$

$$= \frac{1}{4}(\ln 8 - \ln 3) = \frac{1}{4}\ln\left(\frac{8}{3}\right)$$

7.4 #33

$$\int \frac{dx}{x(x^2+4)^2}$$

$$\frac{1}{x(x^2+4)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}$$

$$1 = A(x^2+4)^2 + (Bx+C)(x)(x^2+4) + (Dx+E)x$$

- $x=0$

$1 = 16A$, $\boxed{A = \frac{1}{16}}$

$$1 = \frac{1}{16}(x^2+4)^2 + (Bx+C)(x)(x^2+4) + (Dx+E)x$$

$$16 = (x^4+8x^2+16) + 16(Bx+C)(x^3+4x) + 16(Dx^2+E$$

$$16 = x^4 + 8x^2 + 16 + 16Bx^4 + 64Bx^2 + 16Cx^3 + 64C + 16Dx^2 + 16Ex$$

$$16 = (1+16B)x^4 + (16C)x^3 + (8+64B+16D)x^2 + (64C+16E)x + 16$$

$1 + 16B = 0$

$\boxed{B = \frac{-1}{16}}$

$16C = 0$, $\boxed{C=0}$

$8 + 64B + 16D = 0$

$8 + 64\left(\frac{-1}{16}\right) + 16D = 0$

$8 - 4 + 16D = 0$

$\cancel{\boxed{D=-4}}$ $16D = -4$

$\boxed{D = \frac{-1}{4}}$

$64C + 16E = 0$

$64(0) + 16E = 0$

$\boxed{E = 0}$

$$= \int \frac{1}{16}\cdot\frac{1}{x} + \frac{-1}{16}\cdot\frac{x}{x^2+4} - \frac{1}{4}\cdot\frac{x}{(x^2+4)^2}$$

$$= \frac{1}{16}\ln|x| - \frac{1}{16}\int \frac{x}{x^2+4}\,dx - \frac{1}{4}\int\frac{x}{(x^2+4)^2}$$

$u = x^2+4$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= \frac{1}{16}\ln|x| + \int\left(\frac{-1}{16}\cdot\frac{1}{2}\,\frac{1}{u} - \frac{1}{4}\cdot\frac{1}{2}\,\frac{1}{u^2}\right)du$$

$$= \frac{1}{16}\ln|x| - \frac{1}{32}\ln|u| - \frac{1}{8}\int u^{-2}\,du = \frac{1}{16}\ln|x| - \frac{1}{32}\ln|u| - \frac{1}{8}\frac{u^{-1}}{-1} +$$

$$= \frac{1}{16}\ln|x| - \frac{1}{32}\ln(x^2+4) + \frac{1}{8}\cdot\frac{1}{x^2+4} + C$$

§7.4 #37 $\displaystyle\int \frac{x^2-3x+7}{(x^2-4x+6)^2}\,dx$

$$\frac{x^2-3x+7}{(x^2-4x+6)^2} = \frac{Ax+B}{x^2-4x+6} + \frac{Cx+D}{(x^2-4x+6)^2}$$

$$x^2-3x+7 = (Ax+B)(x^2-4x+6)+(Cx+D)$$

$$x^2-3x+7 = Ax^3-4Ax^2+6Ax+Bx^2-4Bx+6B$$
$$+Cx+D$$

$$x^2-3x+7 = Ax^3+(-4A+B)x^2+(6A-4B+C)x+(6B+D)$$

$\boxed{A=0}$

$(-4A+B)=\cancel{0}\,1$

$-4\cdot 0+B=1$

$\boxed{B=1}$

$6A-4B+C=-3$

$6\cdot 0-4\cdot 1+C=-3$

$-4+C=-3$

$\boxed{C=1}$

$6B+D=7$

$6\cdot 1+D=7$

$\boxed{D=1}$

$$=\int \frac{1}{x^2-4x+6} + \frac{x+1}{(x^2-4x+6)^2}\,dx$$

$$=\int \frac{1}{(x-4x+4)-4+6} + \frac{x+1}{((x-4x+4)-4+6)^2}$$

$$=\int \frac{1}{(x-2)^2+2} + \frac{x+1}{((x-2)^2+2)^2}\,dx$$

$u=x-2$
$du=dx$
$x=u+2$

$$=\int \frac{1}{u^2+2} + \frac{(u+2)+1}{(u^2+2)^2}\,du$$

$$=\int\left(\frac{1}{u^2+2} + \frac{u+3}{(u^2+2)^2}\right)du$$

$$\cancel{=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + \int \frac{u}{(u^2+2)^2}\,du + 3\int \frac{1}{(u^2+2)^2}\,du}$$

#37 cont.

$$= \int \frac{1}{u^2+2}\,du + \int \frac{u}{(u^2+2)^2}\,du + 3\int \frac{1}{(u^2+2)^2}\,du$$

$$\text{I} \quad \text{II} \quad \text{III}$$

I $\int \frac{1}{u^2+2}\,du = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) = \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + C_1$

II $\int \frac{u}{(u^2+2)^2}\,du = \frac{1}{2}\int \frac{dw}{w^2} = \frac{1}{2}\int w^{-2}\,dw$

$w = u^2+2$

$dw = 2u\,du$

$\frac{1}{2}dw = u\,du$

$$= \frac{1}{2}\,\frac{w^{-1}}{-1} + C_2$$

$$= \frac{-1}{2(u^2+2)}$$

$$= \cancel{\frac{1}{2(x^2 \; 4 \; 6}}$$

$$= \frac{-1}{2(x^2-4x+6)} + C_2$$

III $3\int \frac{1}{(u^2+2)^2}\,du$

$u = \sqrt{2}\tan\theta$

$du = \sqrt{2}\sec^2\theta\,d\theta$

$u^2+2 = 2\tan^2\theta + 2 = 2\sec^2\theta$

$$= 3\int \frac{\sqrt{2}\sec^2\theta}{(2\sec^2\theta)^2}\,d\theta = \frac{3\sqrt{2}}{4}\int \frac{1}{\sec^2\theta}\,d\theta$$

$$= \frac{3\sqrt{2}}{4}\int \cos^2\theta\,d\theta$$

$$= \frac{3\sqrt{2}}{4}\cdot\frac{1}{2}\int (1+\cos 2\theta)\,d\theta$$

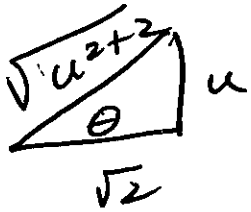
$$= \frac{3\sqrt{2}}{8}\left(\theta + \frac{\sin 2\theta}{2}\right) + C_3$$

$$= \frac{3\sqrt{2}}{8}\theta + \frac{3\sqrt{2}}{16}\sin 2\theta + C_3$$

#37 cont.

$u = \sqrt{2}\tan\theta$

$\tan\theta = \dfrac{u}{\sqrt{2}} \qquad \theta = \tan^{-1}\left(\dfrac{u}{\sqrt{2}}\right)$



$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2\frac{u}{\sqrt{u^2+2}} \cdot \frac{\sqrt{2}}{\sqrt{u^2+2}}$$
$$= \frac{2\sqrt{2}\,u}{u^2+2}$$

$$= \frac{3\sqrt{2}}{8}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + \frac{3\sqrt{2}}{16} \cdot 2\sqrt{2}\,\frac{u}{u^2+2} + C_2$$

$$= \frac{3\sqrt{2}}{8}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{3}{4} \cdot \frac{x-2}{x^2-4x+6} + C_3$$

$$= \mathrm{I} + \mathrm{II} + \mathrm{III}$$

$$= \frac{4\sqrt{2}}{8}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{-2}{4(x^2-4x+6)} + \frac{3\sqrt{2}}{8}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right)$$
$$+ \frac{3}{4}\,\frac{(x-2)}{(x^2-4x+6)}$$

$$= \frac{7\sqrt{2}}{8}\tan^{-1}\left(\frac{x-2}{\sqrt{2}}\right) + \frac{3x-8}{4(x^2-4x+6)} + C$$

§7.4 #39 $\int \frac{1}{x\sqrt{x+1}}\,dx$

$u = \sqrt{x+1}$

$u^2 = x+1, \quad x = u^2 - 1$

$\cancel{2\,du = dx} \quad 2u\,du = dx$

$$\int \frac{1}{(u^2-1)u} \cdot 2u\,du = 2\int \frac{1}{u^2-1}\,du$$

$$\frac{2}{u^2-1} = \frac{2}{(u-1)(u+1)} = \cancel{2}\frac{A}{u-1} + \frac{B}{u+1}$$

$2 = A(u+1) + B(u-1)$

- $u = -1$
  $2 = -2B;$ $\boxed{B = -1}$

- $u = 1$
  $2 = 2A$ $\boxed{A = 1}$

$$\int \left(\frac{1}{u-1} - \frac{1}{u+1}\right)du = \ln|u-1| - \ln|u+1| + C$$

$$= \ln\left|\frac{u-1}{u+1}\right| + C$$

$$= \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C$$

§7.4 #41 $\int_9^{16} \frac{\sqrt{x}}{x-4}\,dx$

$u=\sqrt{x}$

$u^2=x$ *

$2u\,du=dx$

$$\int_3^4 \frac{u}{u^2-4}\cdot 2u\,du = \int_3^4 \frac{2u^2}{u^2-4}\,du$$

$$\begin{array}{r} 2 + 8/u^2-4 \\ u^2-4\,\overline{)\,2u^2+0u+0} \\ -(2u^2 \quad -8) \\ \hline 8 \end{array}$$

$$= \int_3^4 \left(2 + \frac{8}{u^2-4}\right)du$$

$$\frac{8}{u^2-4} = \frac{8}{(u-2)(u+2)} = \frac{A}{u-2} + \frac{B}{u+2}$$

$8 = A(u+2) + B(u-2)$

$8 = (A+B)u + (2A-2B)$

$A+B=0$ $\qquad$ $2A-2B=8$

$A=-B$ $\qquad$ $A-B=4$

$-B-B=4$

$-2B=4$, $\boxed{B=-2}$

$A=-B=-(-2)=2$

$\boxed{A=2}$

$$= \int_3^4 \left(2 + \frac{2}{u-2} - \frac{2}{u+2}\right)du$$

$$= \Big[2u + 2\ln|u-2| - 2\ln|u+2|\Big]_3^4$$

$$= \Big[2(4) + 2\ln|4-2| - 2\ln|4+2|\Big] - \Big[2(3) + 2\ln|3-2| - 2\ln|3+2|\Big]$$

$$= 8 + 2\ln 2 - 2\ln 6 - 6 - 2\ln 1 + 2\ln 5$$

$$= 2 + \ln 4 - \ln 36 + \ln 25 = 2 + \ln\frac{100}{36}$$

$$= 2 + \ln\frac{25}{9}$$

§7.4

43 $\displaystyle\int \frac{x^3}{\sqrt[3]{x^2+1}}\,dx$

$$\boxed{\begin{aligned} u &= \sqrt[3]{x^2+1} \\ u^3 &= x^2+1 \qquad x^2 = u^3-1 \\ 3u^2\,du &= 2x\,dx \\ \tfrac{3}{2}u^2\,du &= x\,dx \end{aligned}}$$

$$= \int \frac{x^2 \cdot}{\sqrt[3]{x^2+1}}\; x\,dx = \int\left(\frac{u^3-1}{u}\right)\cdot \frac{3}{2}u^2\,du$$

$$= \frac{3}{2}\int \frac{(u^3-1)u^2}{u}\,du = \frac{3}{2}\int (u^3-1)u\,du$$

$$= \frac{3}{2}\int (u^4-u)\,du = \frac{3}{2}\left[\frac{u^5}{5} - \frac{u^2}{2}\right] + C$$

$$= \frac{3}{2}\left[\frac{1}{5}\left(\sqrt[3]{x^2+1}\right)^5 - \frac{1}{2}\left(\sqrt[3]{x^2+1}\right)^2\right] + C$$

$$= \frac{3}{10}(x^2+1)^{5/3} - \frac{3}{4}(x^2+1)^{2/3} + C$$

§ 7.4

45 $\displaystyle\int \frac{1}{\sqrt{x} - \sqrt[3]{x}}\, dx$

$u = \sqrt[6]{x}$

$u^6 = x$

$6u^5\,du = dx$

$\sqrt{x} = \sqrt{u^6} = u^3$

$\sqrt[3]{x} = \sqrt[3]{u^6} = u^2$

$$\int \left(\frac{1}{u^3 - u^2}\right) \cdot 6u^5\, du$$

$$= \int \frac{6 \cdot u^5}{u^2(u-1)}\, du = 6\int \frac{u^3}{u-1}\, du$$

$$\begin{array}{r l}
 & u^2 + u + 1 + \frac{1}{u-1} \\
u-1 \,\big) & u^3 + 0u^2 + 0u + 0 \\
 & -(u^3 - u^2) \\
\hline
 & u^2 + 0u \\
 & -(u^2 - u) \\
\hline
 & u + 0 \\
 & -(u - 1) \\
\hline
 & 1
\end{array}$$

$$= 6\int \left[(u^2 + u + 1) + \frac{1}{u-1}\right] du$$

$$= 6\left[\frac{u^3}{3} + \frac{u^2}{2} + u + \ln|u-1|\right] + C$$

$$= 6\left[\frac{\sqrt{x}}{3} + \frac{\sqrt[3]{x}}{2} + \sqrt[6]{x} + \ln|\sqrt[6]{x} - 1|\right] + C$$

$$= 2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + 6\ln|\sqrt[6]{x} - 1| + C$$

§7.4 #47 $\displaystyle\int \frac{e^{2x}}{e^{2x}+3e^x+2}\,dx$

$$= \int \frac{e^{2x}}{(e^x)^2+3(e^x)+2}\,dx$$

$$= \int \frac{e^{2x}}{(e^x+1)(e^x+2)}\,dx$$

$u = e^x$

$du = e^x\,dx$

$$= \int \frac{e^x}{(e^x+1)(e^x+2)} \cdot e^x\,dx$$

$$= \int \frac{u}{(u+1)(u+2)}\,du$$

$$\frac{u}{(u+1)(u+2)} = \frac{A}{u+1} + \frac{B}{u+2}$$

$u = A(u+2) + B(u+1)$

- $u = -2$

  $-2 = -B$, $\boxed{B = 2}$

- $u = -1$

  $-1 = A$ $\boxed{A = -1}$

$$= \int \left(\frac{-1}{u+1} + \frac{2}{u+2}\right) du = -\ln|u+1| + 2\ln|u+2| + C$$

$$= -\ln|u+1| + \ln(u+2)^2 + C$$

$$= \ln\left|\frac{(u+2)^2}{u+1}\right| + C$$

$$= \ln\left[\frac{(e^x+2)^2}{e^x+1}\right] + C$$

§7.4 #49 $\displaystyle\int \frac{\sec^2 t}{\tan^2 t + 3\tan t + 2}\,dt$

$u = \tan t$
$du = \sec^2 t\,dt$

$$\int \frac{du}{u^2+3u+2}\,du = \int \frac{1}{(u+2)(u+1)}\,du$$

$$\frac{1}{(u+2)(u+1)} = \frac{A}{u+2} + \frac{B}{u+1}$$

$1 = A(u+1) + B(u+2)$
$1 = (A+B)u + (A+2B)$

$A+B=0$
$A+2B=1$

$$\begin{array}{r} A+B=0 \\ -A-2B=-1 \\ \hline -B=\cancel{0}-1 \end{array}$$

$\boxed{B=1}$

$A+B=0$
$A=-B$
$\boxed{A=-1}$

$$\int \left(\frac{-1}{u+2} + \frac{1}{u+1}\right) du = -\ln|u+2| + \ln|u+1| + C$$

$$= + \ln\left|\frac{u+1}{u+2}\right| + C$$

$$= \boxed{\ln|\tan t + 1| - \ln|\tan t + 2| + C}$$

$$= + \ln\left|\frac{\tan(t)+1}{\tan(t)+2}\right| + C$$