§7.3 # 1 $\quad \int \frac{1}{x^2\sqrt{x^2-9}}\,dx$

$$x = 3\sec\theta$$
$$dx = 3\sec\theta\tan\theta\,d\theta$$
$$\sqrt{x^2-9} = \sqrt{(3\sec\theta)^2-9}$$
$$= \sqrt{9\sec^2\theta-9} = \sqrt{9(\sec^2\theta-1)}$$
$$= \sqrt{9\tan^2\theta} = 3\tan\theta$$

$$= \int \frac{1}{(3\sec\theta)^2(3\tan\theta)} \cdot 3\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{3\sec\theta\tan\theta}{27\sec^2\theta\tan\theta}\,d\theta = \cancel{\frac{1}{9}\int\frac{1}{\cos\theta}}$$

$$= \int \frac{1}{9}\frac{1}{\sec\theta}\,d\theta = \frac{1}{9}\int\cos\theta\,d\theta = \frac{-1}{9}\sin\theta + C$$

$$x = 3\sec\theta$$
$$\sec\theta = \frac{x}{3}$$
$$\cos\theta = 3/x$$
$$\sin^2\theta = 1-\cos^2\theta = 1-(3/x)^2$$
$$\sin\theta = \sqrt{1-9/x^2} = \sqrt{\frac{x^2}{x^2}-\frac{9}{x^2}} = \frac{\sqrt{x^2-9}}{x}$$

$$= \frac{-1}{9}\frac{\sqrt{x^2-9}}{x} + C$$

§7.3 #3 $\displaystyle\int \frac{x^3}{\sqrt{x^2+9}}\,dx$

$x = 3\tan\theta$
$dx = 3\sec^2\theta\,d\theta$
$\sqrt{x^2+9} = \sqrt{(3\tan\theta)^2+9}$
$= \sqrt{9(\tan^2\theta+1)}$
$= \sqrt{9\sec^2\theta} = 3\sec\theta$

$$\int \frac{(3\tan\theta)^3}{\cancel{3}\sec\theta}\cdot \cancel{3}\sec^2\theta\,d\theta$$

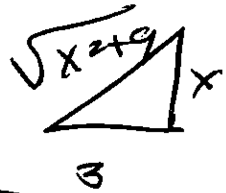
$= 27\int \tan^3\theta\sec\theta\,d\theta$

$= 27\int \tan^2\theta\cdot\tan\theta\sec\theta\,d\theta$
$= 27\int (\sec^2\theta-1)\cdot\tan\theta\sec\theta\,d\theta$

$u = \sec\theta$
$du = \sec\theta\tan\theta\,d\theta$

$= 27\int (u^2-1)\,du = 27\left(\frac{u^3}{3} - u\right) + C$
$= 27\left(\frac{\sec^3\theta}{3} - \sec u\right) + C$

$\tan\theta = \frac{x}{3} = \frac{opp}{adj}$



$\sec\theta = \frac{hyp}{adj} = \frac{\sqrt{x^2+9}}{3} = \frac{(x^2+9)^{1/2}}{3}$

$= 27\left(\frac{(x^2+9)^{3/2}}{3(3)^3} - \frac{\sqrt{x^2+9}}{3}\right) + C$
$= \frac{1}{3}(x^2+9)^{3/2} - 9\sqrt{x^2+9} + C$
$= \frac{1}{3}(x^2+9)\sqrt{x^2+9} - 9\sqrt{x^2+9} + C$
$= \cancel{3x^2+9}\ \frac{1}{3}\sqrt{x^2+9}\left(x^2+9-\frac{27}{\cancel{3}}\right) + C = \frac{1}{3}\sqrt{x^2+9}\,(x^2-18) + C$

§7.3 #5 $\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\,dt$

$t = \sec\theta$

$\sqrt{t^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \tan\theta$

$dt = \sec\theta\tan\theta\,d\theta$

- $t = \sqrt{2}$

  $\sqrt{2} = \sec\theta, \quad \cos\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \quad \theta = \frac{\pi}{4}$

- $t = 2$

  $2 = \sec\theta, \quad \cos\theta = \frac{1}{2}, \quad \theta = \frac{\pi}{3}$

$$\int_{\pi/4}^{\pi/3} \frac{1}{\sec^3\theta\tan\theta}\sec\theta\tan\theta\,d\theta$$

$$= \int_{\pi/4}^{\pi/3} \frac{1}{\sec^2\theta}\,d\theta = \int_{\pi/4}^{\pi/3} \cos^2\theta\,d\theta$$

$$= \int_{\pi/4}^{\pi/3} \frac{1}{2}(1+\cos 2\theta)\,d\theta = \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/4}^{\pi/3}$$

$$= \frac{1}{2}\left[\frac{\pi}{3} + \frac{\sin 2\pi/3}{2}\right] - \frac{1}{2}\left[\frac{\pi}{4} + \frac{\overset{=1}{\sin 2\pi/4}}{2}\right]$$

$$= \frac{1}{2}\left[\pi/3 + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right] - \frac{1}{2}\left[\frac{\pi}{4} + \frac{1}{2}\right]$$

$$= \frac{\pi}{6} + \frac{\sqrt{3}}{8} - \frac{\pi}{8} - \frac{1}{4} = \frac{4}{24}\pi - \frac{3}{24}\pi + \frac{\sqrt{3}}{8} - \frac{2}{8}$$

$$= \frac{\pi}{24} + \frac{\sqrt{3}-2}{8} = \frac{\pi}{24} + \frac{\sqrt{3}}{8} - \frac{1}{4}$$

§7.3 #7 $\displaystyle\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$

$x = 5\sin\theta$

$dx = 5\cos\theta\,d\theta$

$\sqrt{25-x^2} = \sqrt{25-(5\sin\theta)^2}$

$= \sqrt{25(1-\sin^2\theta)}$

$= \sqrt{25\cos^2\theta} = 5\cos\theta$

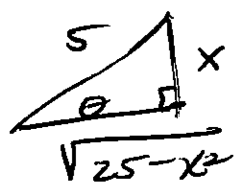
$$= \int \frac{1}{(5\sin\theta)^2\,5\cos\theta}\cdot 5\cos\theta\,d\theta$$

$$= \int \frac{5\cos\theta}{125\sin^2\theta\cos\theta}\,d\theta = \frac{1}{25}\int \frac{1}{\sin^2\theta}\,d\theta$$

$$= \frac{1}{25}\int \csc^2\theta\,d\theta = \frac{1}{25}(-\cot\theta)+C = \frac{-1}{25}\cot\theta + C$$

$x = 5\sin\theta$

$\sin\theta = \frac{x}{5} = \frac{opp}{hyp}$



$\cot\theta = \frac{adj}{hyp}$

$= \frac{\sqrt{25-x^2}}{x}$

$$= \frac{-1}{25}\,\frac{\sqrt{25-x^2}}{x} + C$$

$= \cancel{\frac{-\sqrt{25-x^2}}{25}}$

§7.3 #9 $\displaystyle\int \frac{dx}{\sqrt{x^2+16}}$

$x = 4\tan\theta$

$dx = 4\sec^2\theta\, d\theta$

$\sqrt{x^2+16} = \sqrt{(4\tan\theta)^2+16}$

$= \sqrt{16\tan^2\theta+16}$

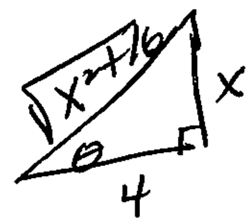
$= \sqrt{16(\tan^2\theta+1)}$

$= \sqrt{16\sec^2\theta} = 4\sec\theta$

$$= \int \frac{4\sec^2\theta\, d\theta}{4\sec\theta} = \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

~~sec~~ $\tan\theta = \dfrac{x}{4} = \dfrac{\text{opp}}{\text{adj}}$



$\sec\theta = \dfrac{\text{hyp}}{\text{adj}} = \dfrac{\sqrt{x^2+16}}{4}$

$$= \boxed{\ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C}$$

put wi C

$= \ln \frac{1}{4}\left|\sqrt{x^2+16} + x\right| + C = \ln\left|\sqrt{x^2+16} + x\right| + \ln\frac{1}{4} + C$

$= \ln\left|\sqrt{x^2+16} + x\right| + C$

§7.3 # 11 $\int \sqrt{1-4x^2}\,dx$

$$x = \tfrac{1}{2}\sin\theta$$

$$dx = \tfrac{1}{2}\cos\theta\,d\theta$$

$$\sqrt{1-4x^2} = \sqrt{1-4(\tfrac{1}{2}\sin\theta)^2}$$

$$= \sqrt{1-4(\tfrac{1}{4})\sin^2\theta} = \sqrt{1-\sin^2\theta}$$

$$= \sqrt{\cos^2\theta} = \cos\theta$$

$$= \int \cos\theta \cdot \tfrac{1}{2}\cos\theta\,d\theta$$

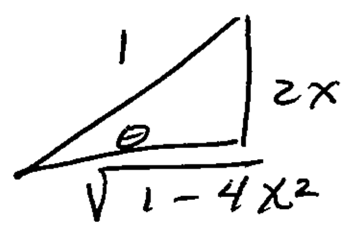
$$= \tfrac{1}{2}\int \cos^2\theta\,d\theta = \tfrac{1}{2}\int \tfrac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= \tfrac{1}{4}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$

$$\sin 2\theta = 2\cos\theta\sin\theta$$

$$x = \tfrac{1}{2}\sin\theta$$

$$\sin\theta = \frac{2x}{1} = \frac{opp}{hyp}$$



$$\cos\theta = \frac{adj}{hyp} = \frac{\sqrt{1-4x^2}}{1} = \sqrt{1-4x^2}$$

$$\sin\theta = 2x$$

$$\theta = \sin^{-1}(2x)$$

$$= \tfrac{1}{4}\left[\sin^{-1}(2x) + \tfrac{1}{2}\cdot 2\sqrt{1-4x^2}\cdot 2x\right] + C$$

$$= \tfrac{1}{4}\sin^{-1}(2x) + \tfrac{1}{2}x\sqrt{1-4x^2} + C$$

§7.3 #13 $\displaystyle\int \frac{\sqrt{x^2-9}}{x^3}\,dx$

$$x = 3\sec\theta$$
$$dx = 3\sec\theta\tan\theta\,d\theta$$

$$\sqrt{x^2-9} = \sqrt{(3\sec\theta)^2-9} = \sqrt{9\sec^2\theta-9}$$
$$= \sqrt{3^2(\sec^2\theta-1)}$$
$$= \sqrt{3^2\tan^2\theta} = 3\tan\theta$$

$$= \int \frac{3\tan\theta}{(3\sec\theta)^3}\cdot 3\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{9\tan^2\theta\sec\theta}{27\sec^3\theta}\,d\theta = \frac{1}{3}\int\frac{\tan^2\theta}{\sec^2\theta}\,d\theta$$

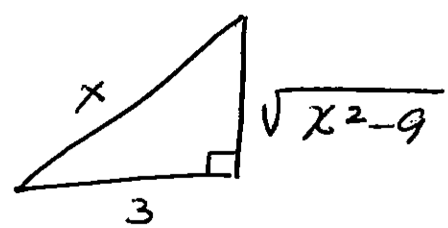
$$= \frac{1}{3}\int \frac{\overset{\sec^2\theta-1}{(\cancel{1-\sec^2\theta})}}{\sec^2\theta}\,d\theta = \frac{1}{3}\int\left(\frac{-1}{\sec^2\theta} + \frac{\sec^2\theta}{\sec^2\theta}\right)d\theta$$

$$= \frac{1}{3}\int(-\cos^2\theta + 1)\,d\theta$$

$$= \frac{1}{3}\int\left[-\frac{1}{2}(1+\cos 2\theta)+1\right]d\theta = \frac{1}{3}\int\left(-\frac{\cos 2\theta}{2} + \frac{1}{2}\right)d\theta$$

$$= \frac{1}{3}\left[-\frac{1}{2}\frac{\sin 2\theta}{2} + \frac{\theta}{2}\right] \cancel{d\theta} + C$$

$$x = 3\sec\theta$$
$$\sec\theta = \frac{x}{3},\quad \boxed{\theta = \sec^{-1}\left(\frac{x}{3}\right)}$$
$$\cos\theta = \frac{3}{x} = \frac{adj}{hyp}$$



$$\sin\theta = \frac{\cancel{opp}}{hyp} = \frac{\sqrt{x^2-9}}{x}$$
$$\sin 2\theta = 2\sin\theta\cos\theta = 2\frac{\sqrt{x^2-9}}{x}\cdot\frac{3}{x} = \frac{6\sqrt{x^2-9}}{x^2}$$

$$= -\frac{1}{12}\sin 2\theta + \frac{1}{6}\theta + C = -\frac{1}{12}\cdot\frac{6\sqrt{x^2-9}}{x^2} + \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) + C$$

$$= \boxed{-\frac{\sqrt{x^2-9}}{2x^2} + \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) + C}$$

§7.3 #15 $\int_0^a x^2\sqrt{a^2-x^2}\,dx$

$x = a\sin\theta$

$dx = a\cos\theta\,d\theta$

$\sqrt{a^2-x^2} = \sqrt{a^2-(a\sin\theta)^2}$

$= \sqrt{a^2-a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)}$

$= \sqrt{a^2\cos^2\theta} = a\cos\theta$

- $x=0$, $0 = a\sin\theta$, $\sin\theta = 0$, $\theta = 0$
- $x=a$, $a = a\sin\theta$, $\sin\theta = 1$, $\theta = \frac{\pi}{2}$

$= \int_0^{\pi/2}(a\sin\theta)^2\cdot a\cos\theta\cdot a\cdot\cos\theta\,d\theta$

$= \int_0^{\pi/2} a^2\sin^2\theta\cdot a^2\cos^2\theta\,d\theta = a^4\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta$

$= a^4\int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{1+\cos 2\theta}{2}\right)d\theta$

$= \frac{a^4}{4}\int_0^{\pi/2}(1-\cos^2 2\theta)\,d\theta = \frac{a^4}{4}\int_0^{\pi/2}\left(1-\frac{1}{2}(1+\cos 4\theta)\right)d\theta$

$= \frac{a^4}{4}\int_0^{\pi/2}\left(\frac{1}{2}-\frac{\cos 4\theta}{2}\right)d\theta = \frac{a^4}{4}\left(\frac{\theta}{2}-\frac{1}{2}\,\frac{\sin 4\theta}{4}\right)\Big|_0^{\pi/2}$

$= \frac{a^4}{4}\left(\frac{(\pi/2)}{2}-\frac{1}{8}\sin 4\left(\frac{\pi}{2}\right)\right) - \frac{a^4}{4}\left(\frac{0}{2}-\frac{1}{8}\sin(4\cdot 0)\right)$

$= \frac{a^4}{4}\left(\frac{1}{2}\cdot\frac{\pi}{2}-\frac{1}{8}\overset{=0}{\sin 2\pi}\right) - 0 = \frac{a^4}{4}\left(\frac{\pi}{4}\right)$

$= \frac{a^4\pi}{16} = \frac{\pi}{16}a^4$

§7.3 #17 $\quad \displaystyle\int \frac{x}{\sqrt{x^2-7}}\,dx$

$$u = x^2 - 7$$
$$du = 2x\,dx$$
$$\tfrac{1}{2}\,du = x\,dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int u^{-1/2}\,du$$

$$= \frac{1}{2}\left(\frac{u^{1/2}}{1/2}\right) + C = \frac{1}{\cancel{2}}\cdot\cancel{2}\,u^{1/2} + C$$

$$= \sqrt{x^2-7} + C$$

§7.3#19 $\int \frac{\sqrt{1+x^2}}{x}\,dx$

$u = \sqrt{1+x^2}$

$u^2 = 1+x^2$

$\cancel{2}u\,du = \cancel{2}x\,dx$

$x^2 = u^2-1$

$$= \int \frac{\sqrt{1+x^2}}{x^2}\,x\,dx = \int \frac{u}{u^2-1}\cdot u\,du$$

$$= \int \frac{u^2}{u^2-1}\,du = \int \left(\frac{u^2-1+1}{u^2-1}\right)du$$

$$= \int \left(\frac{u^2-1}{u^2-1} + \frac{1}{u^2-1}\right)du = \int \left(\overset{\mathrm{I}}{1} + \overset{\mathrm{II}}{\frac{1}{u^2-1}}\right)du$$

I $\int 1\,du = u = \sqrt{1+x^2}$

II $\frac{1}{u^2-1} = \frac{1}{(u-1)(u+1)} = \frac{A}{u-1} + \frac{B}{u+1}$

$1 = A(u+1) + B(u-1)$

- $u=1$, $2A=1$, $A = 1/2$
- $u=-1$, $-2B=1$, $B = -1/2$

$$\int \frac{1}{2}\left(\frac{1}{u-1}\right) + \frac{-1}{2}\left(\frac{1}{u+1}\right)du = \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1|$$

$$= \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| = \frac{1}{2}\ln\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right|$$

$$= \frac{1}{2}\ln\left|\frac{(\sqrt{1+x^2}-1)}{(\sqrt{1+x^2}+1)}\cdot\frac{(\sqrt{1+x^2}-1)}{(\sqrt{1+x^2}-1)}\right| = \frac{1}{2}\ln\left|\frac{(\sqrt{1+x^2}-1)^2}{(1+x^2)-1}\right|$$

$$= \frac{1}{2}\ln\frac{(\sqrt{1+x^2}-1)^2}{(x^2)} = \frac{1}{2}\ln\left|\frac{\sqrt{1+x^2}-1}{x}\right|^2$$

$$= \frac{1}{\cancel{2}}\cdot\cancel{2}\ln\left|\frac{\sqrt{1+x^2}-1}{x}\right|$$

Answer:

$$\sqrt{1+x^2} + \ln\left|\frac{\sqrt{1+x^2}-1}{x}\right| + C$$

§7.3#21) $\int_0^{0.6} \frac{x^2}{\sqrt{9-25x^2}}\,dx$

$x = \frac{3}{5}\sin\theta$

$dx = \frac{3}{5}\cos\theta\,d\theta$

$\sqrt{9-25x^2} = \cancel{\sqrt{9-25\left(\frac{3}{5}\sin\theta\right)^2}}$

$= \sqrt{9-25\left(\frac{3}{5}\sin\theta\right)^2}$

$= \sqrt{9 - \cancel{25}\cdot\frac{9}{\cancel{25}}\sin^2\theta}$

$= \sqrt{9(1-\sin^2\theta)} = \sqrt{3^2\cos^2\theta}$

$= 3\cos\theta$

- $x=0$, $0 = \frac{3}{5}\sin\theta$
  $\sin\theta = 0$
  $\theta = 0$
- $x = 0.6$
  $.6 = 3/5\sin\theta$, $\sin\theta = 1$
  $\theta = \pi/2$

$\int_0^{\pi/2} \frac{(\frac{3}{5}\sin\theta)^2}{3\cos\theta}\cdot\frac{3}{5}\cos\theta\,d\theta$

$= \int_0^{\pi/2} \frac{9/25\cdot\sin^2\theta\cdot\frac{3}{5}\cos\theta}{3\cos\theta}\,d\theta$

$= \frac{27}{125}\cdot\frac{1}{3}\int_0^{\pi/2}\sin^2\theta\,d\theta = \frac{9}{25}\int_0^{\pi/2}\frac{1}{2}(1-\cos 2\theta))\,d\theta$

$= \frac{9}{125}\cdot\frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$

$= \frac{9}{250}\left[\frac{\pi}{2} - \frac{1}{2}\underbrace{\sin 2(\pi/2)}_{=0}\right] - \frac{9}{250}\left[0 - \frac{\overbrace{\sin 2\cdot 0}^{=0}}{2}\right]$

$\cancel{= \frac{9\pi}{500}}$ $\cancel{\frac{9\pi}{500}}$

$= \frac{9\pi}{500}$

§7.3 # 23 $\int \sqrt{5+4x-x^2}\,dx$

$$-x^2+4x+5 = -(x^2-4x+4)+4+5 = -(x-2)^2+9$$

$$\int \sqrt{-(x-2)^2+9}\,dx \qquad \begin{aligned} u &= x-2 \\ du &= dx \end{aligned}$$

$$= \int \sqrt{9-u^2}\,du$$

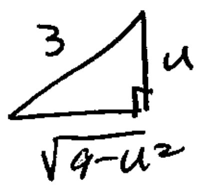
$$u = 3\sin\theta$$
$$du = 3\cos\theta\,d\theta$$

$$\sqrt{9-u^2} = \sqrt{9-(3\sin\theta)^2} = \sqrt{9(1-\sin^2\theta)} = \sqrt{9\cos^2\theta} = 3\cos\theta$$

$$\int 3\cos\theta \cdot 3\cos\theta\,d\theta = 9\int \cos^2\theta\,d\theta$$

$$= 9\cdot\int \frac{1}{2}(1+\cos 2\theta)\,d\theta = \frac{9}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]+C$$

$$u = 3\sin\theta$$
$$\sin\theta = \frac{u}{3}$$



$$\cos\theta = \frac{\sqrt{9-u^2}}{3}$$

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\,\frac{u}{3}\,\frac{\sqrt{9-u^2}}{3}$$

$$\sin\theta = \frac{u}{3}$$
$$\theta = \sin^{-1}\left(\frac{u}{3}\right)$$

$$= \frac{9}{2}\left[\sin^{-1}\left(\frac{u}{3}\right) + \frac{1}{2}\cdot 2\,\frac{u\sqrt{9-u^2}}{9}\right]+C$$

$$= \frac{9}{2}\left[\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{(x-2)\sqrt{5+4x-x^2}}{9}\right]+C$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right) + \frac{(x-2)\sqrt{5+4x-x^2}}{2} + C$$

§7.3 # 25 $\displaystyle\int \frac{x}{\sqrt{x^2+x+1}}\,dx$

$$x^2+x+1 = \left(x^2+x+\tfrac{1}{4}\right)-\tfrac{1}{4}+1$$
$$=(x+1/2)^2+3/4$$
$$u=x+1/2\,,\quad x=u-1/2$$
$$du=dx$$

$$\int \frac{x}{\sqrt{x^2+x+1}}\,dx = \int \frac{x}{\sqrt{(x+1/2)^2+3/4}}\,dx$$
$$= \int \frac{(u-1/2)}{\sqrt{u^2+3/4}}\,du = \int \frac{u}{(u^2+3/4)^{1/2}}\,du + \int \frac{-1/2}{\sqrt{u^2+3/4}}\,du$$
$\quad\text{I} \qquad\qquad \text{II}$

I $\displaystyle\int \frac{u}{(u^2+3/4)^{1/2}}\,du = \frac{1}{2}\int \frac{dw}{w^{1/2}} = \frac{1}{2}\int w^{-1/2}\,dw$

$w=u^2+3/4$

$dw=2u\,du$

$\frac{1}{2}dw=u\,du$

$$=\frac{1}{2}\,\frac{w^{1/2}}{1/2} = w^{1/2} = \sqrt{u^2+3/4}$$
$$=\sqrt{x^2+x+1}$$

II $\displaystyle -\frac{1}{2}\int \frac{1}{\sqrt{u^2+3/4}}\,du$

$\frac{\sqrt{3}}{2}\tan\theta = u \qquad \sqrt{u^2+3/4} = \sqrt{\frac{3}{4}(\tan^2\theta+1)} = \sqrt{\frac{3}{4}\sec^2\theta} = \frac{\sqrt{3}}{2}\sec\theta$

$\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta = du$

$$= -\frac{1}{2}\int \frac{1}{\frac{\sqrt{3}}{2}\sec\theta}\cdot\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$$
$$= -\frac{1}{2}\int \sec\theta\,d\theta = -\frac{1}{2}\ln|\sec\theta+\tan\theta|$$

$\tan\theta = \frac{2u}{\sqrt{3}} = \frac{opp}{adj} \qquad \sec\theta = \sqrt{\tan^2\theta+1} = \sqrt{\left(\frac{2u}{\sqrt{3}}\right)^2+1}$

$= \frac{2(x+1/2)}{\sqrt{3}} = \frac{2x+1}{\sqrt{3}}$

$$=\sqrt{\frac{4u^2}{3}+1}$$
$$=\frac{2}{\sqrt{3}}\sqrt{u^2+3/4}$$
$$\sec\theta = \frac{2}{\sqrt{3}}\sqrt{x^2+x+1}$$

$$= -\frac{1}{2}\ln\left|\frac{2}{\sqrt{3}}\sqrt{x^2+x+1}+\frac{2x+1}{\sqrt{3}}\right|$$

$$\cancel{= \sqrt{x^2+x+1} - \frac{1}{2}\ln\left|\frac{2}{\sqrt{3}}\sqrt{x^2+x+1}+\frac{2x+1}{\sqrt{3}}\right| + C}$$
$$\cancel{= \sqrt{\;}}$$

§ 7.3 #25 cont.

$$= -\frac{1}{2} \ln\left| \frac{2}{\sqrt{3}} \left(\sqrt{x^2+x+1} + x + \frac{1}{2} \right) \right|$$

$$= -\frac{1}{2}\left(\ln\left| \sqrt{x^2+x+1} + x + \frac{1}{2} \right| + \ln \frac{2}{\sqrt{3}} \right)$$

put with constant

Answer:

$$\sqrt{x^2+x+1} - \frac{1}{2} \ln\left| \sqrt{x^2+x+1} + x + \frac{1}{2} \right| + C$$

§7.3 #27 $\int \sqrt{x^2+2x}\, dx$

$= \int \sqrt{(x^2+2x+1)-1}\, dx$

$= \int \sqrt{(x+1)^2-1}\, dx$ $\qquad$ $u = x+1$, $du = dx$

$= \int \sqrt{u^2-1}\, du$

$u = \sec\theta$

$du = \sec\theta \tan\theta\, d\theta$

$\sqrt{u^2-1} = \sqrt{\sec^2\theta - 1}$

$= \sqrt{\tan^2\theta} = \tan\theta$

$= \int \tan\theta \cdot \sec\theta \tan\theta\, d\theta$

$= \int \tan^2\theta \sec\theta\, d\theta = \int (\sec^2\theta - 1)\sec\theta\, d\theta$

$= \int (\sec^3\theta - \sec\theta)\, d\theta$

$= \int \sec^3\theta\, d\theta - \int \sec\theta\, d\theta$ $\leftarrow$ These integrals were solved in §7.2

$= \left(\frac{1}{2}\sec\theta \tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right) - \left(\ln|\sec\theta + \tan\theta|\right) + C$

$u = \sec\theta$, $\sec\theta = u = x+1$

$\tan^2\theta + 1 = \sec^2\theta = u^2$

$\tan^2\theta = u^2 - 1$

$\tan\theta = \sqrt{u^2-1} = \sqrt{x^2+2x}$

$= \frac{1}{2}(x+1)\sqrt{x^2+2x} - \frac{1}{2}\ln\left|x+1+\sqrt{x^2+2x}\right| + C$

§7.3 #29 $\int x\sqrt{1-x^4}\,dx$

$$\int x\sqrt{1-(x^2)^2}\,dx$$

$u = x^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= \frac{1}{2}\int\sqrt{1-u^2}\,du$$

$u = \sin\theta$

$\sqrt{1-u^2} = \sqrt{1-\sin^2\theta}$

$= \sqrt{\cos^2\theta} = \cos\theta$

$du = \cos\theta\,d\theta$

$$= \frac{1}{2}\int\cos\theta\cos\theta\,d\theta = \frac{1}{2}\int\cos^2\theta\,d\theta$$

$$= \frac{1}{2}\int\frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$

$$= \frac{1}{4}\left[\theta + \frac{2\sin\theta\cos\theta}{2}\right] + C$$

$\theta = \sin^{-1}(u)$

$u = \sin\theta$, $\sin^2\theta = 1-u^2$

$\cos^2\theta = 1-\sin^2\theta = 1-u^2$

$\cos\theta = \sqrt{1-u^2}$

$$= \frac{1}{4}\left[\sin^{-1}(u) + u\sqrt{1-u^2}\right] + C$$

$$= \frac{1}{4}\left[\sin^{-1}(x^2) + x^2\sqrt{1-(x^2)^2}\right] + C$$

$$= \frac{1}{4}\sin^{-1}(x^2) + \frac{1}{4}x^2\sqrt{1-x^4} + C$$