

Lecture Notes

Math 185, Calculus II

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7 Techniques of Integration

7.1 Integration by Parts

Integration by Parts is a method of integration. We can derive the formula by first taking the derivative of the product of functions $u(x)v(x)$.

$$(u(x)v(x))' = u(x)'v(x) + u(x)v(x)'$$

We rearrange the terms.

$$(u(x)v(x))' - u(x)'v(x) = u(x)v(x)'$$

$$u(x)v(x)' = (u(x)v(x))' - v(x)u(x)'$$

Now integrate both sides.

$$\int u(x)v(x)' dx = \int ((u(x)v(x))' - v(x)u(x)') dx$$

$$\int u(x)v(x)' dx = u(x)v(x) - \int v(x)u(x)' dx$$

We can simplify the formula by writing

- u for $u(x)$
- du for $u'(x) dx$
- v for $v(x)$
- dv for $v'(x) dx$

<p>Formula for Integration by Parts</p> $\int u dv = uv - \int v du$

The integral of a polynomial times $\sin x$, $\cos x$, or e^x is typical case where integration by parts is used. This type of integral will certainly be on the chapter test.

Example 1 Find $\int x \sin x \, dx$.

Example 2 Find $\int \ln x \, dx$.

Example 3 Find $\int t^2 e^t dx$.

This is a problem that will certainly be on the chapter test.

Example 4 Evaluate $\int e^x \sin x \, dx$.

Tabular Integration is a cheap trick to evaluate integrals like

$$\int x^3 \sin x \, dx.$$

Differentiate	x^3	+	$\sin x$	Integrate
↓	$3x^2$	-	$-\cos x$	↓
	$6x$	+	$-\sin x$	
	6	-	$\cos x$	
	0	/	$\sin x$	

We get the answer

$$\begin{aligned} \int x^3 \sin x \, dx &= +(x^3)(-\cos x) - (3x^2)(-\sin x) + (6x)(\cos x) - (6)(\sin x) + C \\ &= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C \end{aligned}$$

Example Evaluate using tabular integration.

$$\int x^2 e^{3x} \, dx$$

The formula for integration by parts can be used for definite integrals as well.

$$\int_a^b u \, dv = [uv]_a^b - \int_a^b v \, du$$

Example 5 Calculate $\int_0^1 \tan^{-1}(x) \, dx$.

A **reduction formula** is a formula that does not solve an integral, but takes it one more step towards the solution.

Example 6 Prove the reduction formula:

$$\int \sin^n x \, dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x \, dx,$$

where $n \geq 2$ is an integer.

SOLUTION

Let $u = \sin^{n-1} x$ and $dv = \sin x \, dx$. Integration by parts gives

Example Evaluate the given integral using the reduction formula.

$$\int \sin^4 x \, dx$$

Note that another way to solve this integral would be to use the identity $\sin^2 x = \frac{1 - \cos 2x}{2}$.

7.2 Trigonometric Integrals

In this section we integrate certain combinations of trigonometric functions using the substitution method and trigonometric identities.

Powers of Cosine and Sine

Example 1 Evaluate $\int \cos^3 x \, dx$.

Example 2 Find $\int \sin^5 x \cos^2 x \, dx$.

Example 3 Evaluate $\int_0^\pi \sin^2 x \, dx$.

Example 4 Find $\int \sin^4 x \, dx$.

Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

1. If the power of cosine is odd, save the cosine factor and use $\cos^2 x = 1 - \sin^2 x$ and then let $u = \sin x$.
2. If the power of sine is odd, save the sine factor and use $\sin^2 x = 1 - \cos^2 x$ and then let $u = \cos x$.
3. If both the powers of sine and cosine are even, then use the identities

$$\cos^2 u = \frac{1 + \cos 2u}{2} \quad \sin^2 x = \frac{1 - \cos 2x}{2}$$

You also might use $\sin x \cos x = \frac{1}{2} \sin 2x$.

An entirely different method would be to use the reduction formula given in §7.1.

Powers of Secant and Tangent

Example 5 Evaluate $\int \tan^6 x \sec^4 x \, dx$.

Example 6 Find $\int \tan^5 x \sin^7 x \, dx$.

Strategy for Evaluating $\int \tan^m x \sec^n x \, dx$.

1. If the power of secant is even, save a $\sec^2 x$ factor and use $\sec^2 x = 1 + \tan^2 x$ and use $u = \tan x$.
2. If the power of tangent is odd, save a $\sec x \tan x$ factor and use $\tan^2 x = \sec^2 x - 1$ and use $u = \sec x$.

For other cases, there might not be single method to use.

Example Prove the formula:

$$\int \tan x \, dx = \ln |\sec x| + C$$

Example Prove the formula:

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C$$

Hint: Multiply the numerator and denominator of integrand by $\sec x + \tan x$.

Example 7 Find $\int \tan^3 x \, dx$.

SOLUTION

$$\begin{aligned}\int \tan^3 x \, dx &= \int \tan x \tan^2 x \, dx \\ &= \int \tan x (\sec^2 x - 1) \, dx\end{aligned}$$

Finish this work please.

Example 8 Find $\int \sec^3 x \, dx$.

For homework §7.1 #50, we are asked to prove the reduction formula.

$$\sec^n x = \frac{\tan x \sec^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx \quad (n \neq 1)$$

We can use this formula here, or we can follow the same steps, using integration by parts, that we used to prove the reduction formula.

We can use the following trigonometric identities to solve integrals.

To evaluate integrals of the form (a) $\int \sin mx \cos nx \, dx$,
(b) $\int \sin mx \sin nx \, dx$, or (c) $\int \cos mx \cos nx \, dx$, use

$$\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$$

$$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$$

Example 9 Evaluate $\int \sin 4x \cos 5x \, dx$.

7.3 Trigonometric Substitutions

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2 \theta = \cos^2 \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2 \theta = \sec^2 \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta, \quad 0 \leq \theta < \frac{\pi}{2} \text{ or } \pi \leq \theta < \frac{3\pi}{2}$	$\sec^2 \theta - 1 = \tan^2 \theta$

Example 1 Evaluate

$$\int \frac{\sqrt{9 - x^2}}{x^2} dx$$

Example 2 Find $\int \frac{1}{x^2\sqrt{x^2+4}} dx$.

Example 3 Find $\int \frac{x}{\sqrt{x^2 + 4}} dx$.

Example 4 Evaluate $\int \frac{dx}{\sqrt{x^2 - a^2}}$, where $a \geq 0$.

Example 5 Find $\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2 + 9)^{3/2}} dx$.

Example 6 Evaluate $\int \frac{x}{\sqrt{3 - 2x - x^2}} dx$.

7.4 Integration of Rational Functions by Partial Fractions and Rationalizing Substitution

In this section, we study the integration of rational functions. A function is a **rational function** if it is a fraction of polynomials. A polynomial is called **linear** if it is of the form $ax + b$, where a and b are constants. A polynomial is a **quadratic** if it can be written in the form $ax^2 + bx + c$, where a, b, c are real numbers. Every polynomial has a factorization over the real numbers where the factors are either linear or quadratic.

To help explain the method of partial fractions, we start by adding to two rational functions.

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{x+5}{x^2+x-2}$$

Suppose we are asked to solve the integral $\int \frac{x+5}{x^2+x-2} dx$. We can first try u -substitution by letting u equal the denominator. We see that this goes nowhere.

Instead, we write the partial fraction decomposition of the rational function.

$$\int \frac{x+5}{x^2+x-2} dx = \int \frac{2}{x-1} dx - \int \frac{1}{x+2} dx$$

The expression on the right hand side is easy to evaluate.

$$= 2 \ln|x-2| - \ln|x+2| + C$$

We only want to look at rational functions where the denominator has greater degree than the numerator. If the degree of the denominator is not less than that of the numerator, then the rational function is called **improper**. If the rational function is improper, then our first we need to do long division.

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Example 1 Find $\int \frac{x^3 + x}{x - 1} dx$.

Now assume that we have a rational function that is not improper. Assume that we have an integral of the form

$$\int \frac{P(x)}{Q(x)} dx$$

where $\deg P < \deg Q$.

A polynomial with real coefficients can be written as the product of linear factors $ax + b$ and irreducible quadratic factors $ax^2 + bx + c$, $b^2 - 4ac < 0$. The rational function can then be expressed as a sum of partial fractions. Here are the steps we should follow:

1. If the rational function is improper, do long division to write the function as a sum of a polynomial and a proper rational function.
2. For the proper rational function, factor the denominator.
3. Write the proper fraction as a sum of partial fractions.

To perform the last step, we have four separate cases.

Case I: The denominator $Q(x)$ is a product of distinct linear factors.

$$Q(x) = (ax_1 + b_1)(a_2x + b_2) \cdots (a_kx + b_k)$$

Then we can write an expansion

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}$$

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Example 2 Evaluate $\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} dx$.

Example 3 Find $\int \frac{dx}{x^2 - a^2}$, $a \neq 0$.

Case II: $Q(x)$ is a product of linear factors, some of which are repeated.

If a linear factor is repeated r times, then instead of using $A_1/(a_1x + b_1)$, we use

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}$$

Example 4 : Find $\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} dx$.

Case III: $Q(x)$ contains irreducible quadratic factors, none of which is repeated.

The term involving the quadratic factor can be written as

$$\frac{Ax + B}{ax^2 + bx + c}$$

For example,

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{x^2+4}$$

To integrate the last two terms, use the following formula.

Formula:

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

Example 5 Evaluate $\frac{2x^2 - x + 4}{x^3 + 4x} dx$.

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A quadratic is called **irreducible** if it cannot be factored into two linear factors over the real numbers. The **discriminant** of a quadratic $ax^2 + bx + c$ is the expression $b^2 - 4ac$. When we write out the quadratic formula, we see that this is the expression under the radical. If the discriminant is negative, then the quadratic does not have real roots, and is therefore irreducible.

If the rational function has a denominator that is an irreducible quadratic of the form $ax^2 + bx + c$, then we can complete the square.

Example Evaluate $\int \frac{4x - 5}{x^2 + 4x + 13} dx$.

Case IV: $Q(x)$ contains repeated irreducible quadratic factors.

If $Q(x)$ has the factor $(ax^2 + bx + c)^r$, $b^2 - 4ac < 0$, then use the expansion

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

Example 7 : Write the partial fraction expansion of the following.

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

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Example 8 Evaluate $\int \frac{1 - x + 2x^2 - x^3}{x(x^2 + 1)^2} dx$.

Rationalizing Substitutions

We are now going to look at integrating functions that contain radical expressions, or nonrational expressions. If a function contains nonrational expressions, then we use u -substitution. We let u equal the radical.

Example 9 Evaluate $\int \frac{\sqrt{x+4}}{x} dx$.