Lecture Notes
Math 185, Calculus II

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Contents

7 Techniques of Integration

7.1 Integration by Parts

Integration by Parts is a method of integration. We can derive the formula by first takeing the derivative of the product of functions $u(x)v(x)$.

$$(u(x)v(x))' = u(x)'v(x) + u(x)v(x)'$$

We rearrange the terms.

$$(u(x)v(x))' - u(x)'v(x) = u(x)v(x)'$$

$$u(x)v(x)' = (u(x)v(x))' - v(x)u(x)'$$

Now integrate both sides.

$$\int u(x)v(x)'\ dx = \int \left((u(x)v(x))' - v(x)u(x)'\right)\ dx$$

$$\int u(x)v(x)'\ dx = u(x)v(x) - \int v(x)u(x)'\ dx$$

We can simplify the formula by writing

- u for $u(x)$
- du for $u'(x)\ dx$
- v for $v(x)$
- dv for $v'(x)\ dx$

Formula for Integration by Parts

$$\int u\ dv = uv - \int v\ du$$

The integral of a polynomial times $\sin x$, $\cos x$, or e^x is typical case where integration by parts is used. This type of integral will certainly be on the chapter test.

Example. Find $\int x \sin x\ dx$.

Example. Find $\int \ln x\ dx$.

Example. Find $\int t^2 e^t\ dx$.

This is a problem that will certainly be on the chapter test.

Example. Evaluate $\int e^x \sin x\ dx$.

Tabular Integration is a cheap trick to evaluate integrals like

$$\int x^3 \sin x\ dx.$$

Differentiate			Integrate
x^3	$+$	$\sin x$	
$3x^2$	$-$	$-\cos x$	
$6x$	$+$	$-\sin x$	
6	$-$	$\cos x$	
0		$\sin x$	

We get the answer

$$\begin{aligned}\int x^3 \sin x \; dx &= +(x^3)(-\cos x) - (3x^2)(-\sin x) + (6x)(\cos x) - (6)(\sin x) + C \\ &= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C\end{aligned}$$

Example. Evaluate using tabular integration.

$$\int x^2 e^{3x} \; dx$$

The formula for integration by parts can be used for definite integrals as well.

$$\int_a^b u \; dv = [uv]_a^b - \int_a^b v \; du$$

Example. Calculate $\int_0^1 \tan^{-1}(x) \; dx$.

A **reduction formula** is a formula that does not solve an integral, but takes it one more step towards the solution.

Example. Prove the reduction formula:

$$\int \sin^n x \; dx = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x \; dx,$$

where $n \geq 2$ is an integer.

SOLUTION

Let $u = \sin^{n-1} x$ and $dv = \sin x \; dx$. Integration by parts gives

Example. Evaluate the given integral using the reduction formula.

$$\int \sin^4 x \; dx$$

Note that another way to solve this integral would be to use the identity $\sin^2 x = \dfrac{1 - \cos 2x}{2}$.

7.2 Trigonometric Integrals

In this section we integrate certain combinations of trigonometric functions using the substitution method and trigonometric identities.

Powers of Cosine and Sine

Example. Evaluate $\int \cos^3 x \; dx$.

Example. Find $\int \sin^5 x \cos^2 x \; dx$.

Example. Evaluate $\int_0^{\pi} \sin^2 x \; dx$.

Example. Find $\int \sin^4 x \; dx$.

Strategy for Evaluating $\int \sin^m x \cos^n x \; dx$

1. If the power of cosine is odd, save the cosine factor and use $\cos^2 x = 1 - \sin^2 x$ and then let $u = \sin x$.
2. If the power of sine is odd, save the sine factor and use $\sin^2 x = 1 - \cos^2 x$ and then let $u = \cos x$.
3. If both the powers of sine and cosine are even, then use the identities

$$\cos^2 u = \frac{1 + \cos 2u}{2} \qquad \sin^2 x = \frac{1 - \cos 2x}{2}$$

You also might use $\sin x \cos x = \frac{1}{2} \sin 2x$.

An entirely different method would be to use the reduction formula given in §7.1.

Powers of Secant and Tangent

Example. Evaluate $\int \tan^6 x \sec^4 x \; dx$.

Example. Find $\int \tan^5 x \sin^7 x \; dx$.

Strategy for Evaluating $\int \tan^m x \sec^n x \ dx$.

1. If the power of secant is even, save a $\sec^2 x$ factor and use $\sec^2 x = 1 + \tan^2 x$ and use $u = \tan x$.
2. If the power of tangent is odd, save a $\sec x \tan x$ factor and use $\tan^2 x = \sec^2 x - 1$ and use $u = \sec x$.

For other cases, there might not be single method to use.

Example. Prove the formula:

$$\int \tan x \ dx = \ln|\sec x| + C$$

Example. Prove the formula:

$$\int \sec x \ dx = \ln|\sec x + \tan x| + C$$

Hint: Multiply the numerator and denominator of integrand by $\sec x + \tan x$.

Example. Find $\int \tan^3 x \ dx$.

Solution

$$\begin{aligned} \int \tan^3 x \ dx &= \int \tan x \tan^2 x \ dx \\ &= \int \tan x (\sec^2 x - 1) \ dx \end{aligned}$$

Finish this work please.

Example. Find $\int \sec^3 x \ dx$.

For homework §7.1 #50, we are asked to prove the reduction formula.

$$\sec^n x = \frac{\tan x \sec^{n-1} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \ dx \quad (n \neq 1)$$

We can use this formula here, or we can follow the same steps, using integration by parts, that we used to prove the reduction formula.

We can use the following trigonometric identities to solve integrals.

To evaluate integrals of the form (a)$\int \sin mx \cos nx \ dx$, (b) $\int \sin mx \sin nx \ dx$, or (x)$\int \cos mx \cos nx \ dx$, use

$$\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$$

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$

Example. Evaluate $\int \sin 4x \cos 5x \ dx$.

7.3 Trigonometric Substitutions

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta, \quad 0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Example. Evaluate

$$\int \frac{\sqrt{9 - x^2}}{x^2} dx$$

Example. Find $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\ dx$.

Example. Find $\displaystyle\int \frac{x}{\sqrt{x^2+4}}\ dx$.

Example. Evaluate $\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}}$, where $a \geq 0$.

Example. Find $\displaystyle\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\ dx$.

Example. Evaluate $\displaystyle\int \frac{x}{\sqrt{3 - 2x - x^2}}\ dx$.

7.4 Integration of Rational Functions by Partial Fractions and Rationalizing Substitution

In this section, we study the integration of rational functions. A function is a **rational function** if it is a fraction of polynomials. A polynomial is called **linear** if it is of the form $ax + b$, where a and b are constants. A polynomial is a **quadratic** if it can be written in the form $ax^2 + bx + c$, where a, b, c are real numbers. Every polynomial has a factorization over the real numbers where the factors are either linear or quadratic.

To help explain the method of partial fractions, we start by adding to two rational functions.

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{x+5}{x^2+x-2}$$

Suppose we are asked to solve the integral $\int \frac{x+5}{x^2+x-2}\,dx$. We can first try u-substitution by letting u equal the denominator. We see that this goes nowhere.

Instead, we write the partial fraction decomposition of the rational function.

$$\int \frac{x+5}{x^2+x-2}\,dx = \int \frac{2}{x-1}\,dx - \int \frac{1}{x+2}\,dx$$

The expression on the right hand side is easy to evaluate.

$$= 2\ln|x-2| - \ln|x+2| + C$$

We only want to look at rational functions where the denominator has greater degree than the numerator. If the degree of the denominator is not less than that of the numerator, then the rational function is called **improper**. If the rational function is improper, then our first we need to do long division.

Example. Find $\int \frac{x^3+x}{x-1}\,dx$.

Now assume that we have a rational function that is not improper. Assume that we have an integral of the form

$$\int \frac{P(x)}{Q(x)}\,dx$$

where $\deg P < \deg Q$.

A polynomial with real coefficients can be written as the product of linear factors $ax + b$ and irreducible quadratic factors $ax^2 + bx + c$, $b^2 - 4ac < 0$. The rational function can then be expressed as a sum of partial fractions. Here are the steps we should follow:

1. If the rational function is improper, do long division to write the function as a sum of a polynomial and a proper rational function.
2. For the proper rational function, factor the denominator.
3. Write the proper fraction as a sum of partial fractions.

To perform the last step, we have four separate cases.

Case I: The denominator $Q(x)$ is a product of distinct linear factors.

$$Q(x) = (ax_1 + b_1)(a_2x + b_2)\cdots(a_kx + b_k)$$

Then we can write an expansion

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}$$

Example. Evaluate $\displaystyle\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\, dx$.

Example. Find $\displaystyle\int \frac{dx}{x^2 - a^2}\ dx, \quad a \neq 0$.

Case II: $Q(x)$ is a product of liner factors, some of which are repeated.

If a linear factor is repeated r times, then instead of using $A_1/(a_1x + b_1)$, we use

$$\frac{A_1}{a_1x + b_1} + \frac{A_2}{(a_1x + b_1)^2} + \cdots + \frac{A_r}{(a_1x + b_1)^r}$$

Example. : Find $\displaystyle\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx$.

Case III: $Q(x)$ contains irreducible quadratic factors, none of which is repeated.

The term involving the quadratic factor can be written as

$$\frac{Ax + B}{ax^2 + bx + c}$$

For example,

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx + C}{x^2+1} + \frac{Dx+E}{x^2+4}$$

To integrate the last two terms, use the following formula.

Formula:

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Example. Evaluate $\dfrac{2x^2 - x + 4}{x^3 + 4x}\,dx$.

A quadratic is called **irreducible** if it cannot be factored into two linear factors over the real numbers. The **discriminant** of a quadratic $ax^2 + bx + c$ is the expression $b^2 - 4ac$. When we write out the quadratic formula, we see that this is the expression under the radical. If the discriminant is negative, then the quadratic does not have real roots, and is therefore irreducible.

If the rational function has a denominator that is an irreducible quadratic of the form $ax^2 + bx + c$, then we can complete the square.

Example. Evaluate $\displaystyle\int \frac{4x-5}{x^2+4x+13}\,dx$.

Case IV: $Q(x)$ contains repeated irreducible quadratic factors.

If $Q(x)$ has the factor $(ax^2+bx+c)^r$, $b^2-4ac < 0$, then use the expansion

$$\frac{A_1x + B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots \frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

Example. : Write the partial fraction expansion of the following.

$$\frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

Example. Evaluate $\displaystyle\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2}\,dx$.

Rationalizing Substitutions

We are now going to look at integrating functions that contain radical expressions, or nonrational expressions. If a functions contains nonrational expressions, then we use u-substitution. We let u equal the radical.

Example. Evaluate $\int \frac{\sqrt{x+4}}{x}\, dx$.

7.7 Approximate Integration

- For certain definite integrals, it is not possible to find an antiderivative. For example, the following:

$$\int_0^1 e^{x^2}\ dx$$

and

$$\int_{-1}^1 \sqrt{1+x^3}\ dx$$

Also, if a function is defined by discrete data points through an experiment, then we will not be able to find an antiderivative.

- We can use Reimann sums to approximate integrals.

Recall that a Reimann sum is of the form

$$\sum_{i=1}^{n} f(x_i^*)\ \Delta x$$

where x_i^* is any value in the interval $[x_{i-1}, x_1]$. We can use the following to approximate the definite integral $\int_a^b f(x)\ dx$.

  - Left endpoint approximation:

$$L_n = \sum_{i=1}^{n} f(x_{i-1}\ \Delta x$$

  - Right endpoint approximation:

$$R_n = \sum_{i=1}^{n} f(x_i)\ \Delta x$$

  - Midpoint Rule:

$$M_n = \sum_{i=1}^{n} f(\overline{x_i})\ \Delta x$$

  where $\overline{x_i} = \frac{x_{i-1}+x_1}{2}$.

- Trapezoid Rule: Another approximation method is the Trapezoid Rule. This is found by averaging L_n and R_n. Pictorially this looks like a linear approximation of the curve f and forms trapezoids underneath which we calculate the area.

  $$T_n = \frac{L_n + R_n}{2} = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

  Here $\Delta x = (b - a)/n$ and $x_i = a + i\Delta x$.

- Example: Use (a) the Trapezoid Rule and (b) the Midpoint Rule with $n = 5$ to approximate the integral $\int_1^2 (1/x)\ dx$.

  SOLUTION $T_5 \approx 0.695635$ and $M_n \approx 0.691908$.

  Note that
  $$\int_1^2 (1/x)\ dx = [\ln x]_1^2 = \ln 2 \approx 0693147\ldots$$

- The error in using an approximation is defined to be the difference between the true value and the approximation.

  Define
  $$E_T = \int_a^b f(x)\ dx - T_n$$
  and
  $$E_M = \int_a^b f(x)\ dx - M_n$$

  In the previous example,
  $$E_T \approx -0.002488 \text{ and } E_M \approx 0.001239$$

- Error Bounds. Suppose $|f''(x)| \leq K$ for all $a \leq x \leq b$. If E_T and E_M are the errors in the Trapezoid and Midpoint Rules, then

  $$|E_T| \leq \frac{K(b-a)^3}{12n^2} \text{ and } |E_M| \leq \frac{K(b-a)^3}{24n^2}$$

  We can conclude that the Midpoint Rule gives a better approximation than the Trapezoid Rule. Error bounds for the left Reimann sum and right Reimann sums are not given here, but these approximation methods can be shown to have a greater error than the Midpoint and Trapezoid Rules.

- Find an upper bound for the errors T_n for the integral $\int_1^2 (1/x)\,dx$.

  Clearly the issue is finding the value K which is the maximum value of $|f''(x)|$ on the interval $[a, b]$.

$$f(x) = 1/x,\ f'(x) = -1/x^2,\ f''(x) = 2/x^3$$

  To find the maximum value of this function, f'', you can take another derivative, find the critical values and compare the functions values at the critical numbers and endpoints. In this case, it suffices to note that $f''(x)$ is decreasing on the interval $[1, 2]$ and takes its maximum value at $x = 1$. Therefore

$$K = f''(1) = 2/(1)^3 = 2.$$

  Note that this is the best value for K. Sometimes we might have to settle for a value of K that is not actually the maximum value of the function.

  We can now use the formulas using $n = 5$.

$$|E_T| \le \frac{K(b-a)^3}{12n^2} = \frac{2(2-1)^3}{12(5)^2} = \frac{1}{150} \approx 0.006667 \quad \square$$

- How large should we take n in order to guarantee that the Trapezoid and Midpoint Rule approximations for $\int_1^2 (1/x)\,dx$ are accurate to within 0.0001?

  SOLUTION

  We want the error less than 0.0001, so if we choose n so that the upper bound on the error is less than this value, we will have done the job.

  We want to solve for n so that

$$|E_T| \le \frac{K(b-a)^3}{12n^2} < 0.0001$$

  In this case, $a = 1,\ b = 2$, and $K = 2$.

  Our problem reduces finding the smallest value n that satisfies

$$\frac{2(2-1)^3}{12n^2} =< 0.0001$$

i.e. $\dfrac{1}{6n^2} < 0.0001$.
It suffices that $n^2 > \frac{1}{6(0.0001)}$, i.e.

$$n > \frac{1}{\sqrt{0.0006}} \approx 40.8$$

Thus $n = 41$ is the smallest value of n that will give the accuracy that is asked for. □

- Example

  1. Use the Midpoint Rule with $n = 10$ to approximate the integral $\int_0^1 e^{x^2}\, dx$.
  2. Give an upper bound for the error involved in this approximation.

  SOLUTION

  1. $M_5 \approx 1.460393$.
  2. The difficult part is finding K.

  $$f(x) = e^{x^2}, \quad f'(x) = 2xe^{x^2}, \quad f''(x) = (2+4x^2)e^{x^2}$$

  The function $f''(x)$ is increasing on $[0, 1]$ so therefore it takes its maximum value at $x = 1$. Therefore $|f''(x) \leq (2 + 4 \cdot 1^2)e^{1^2} = 6e$ on $[0, 1]$. Choosing $K = 6e$ gives an upper bound of

  $$|E_M| \leq \frac{K(b-a)^3}{24n^2} = \frac{6e(1-0)^3}{24(10)^2} = \frac{e}{400} \approx 0.007 \quad \square$$

- **Simpson's Rule**

  Simpson's Rule is based on making a quadratic approximation of the curve given by f. Make partition of n intervals, choose n even. Through every three points on the curve made by the partition, draw a parabola. The area under this parabola can give us an approximation for the area under the curve. We can write an equation for the parabola in terms of the y-values of the three points and then we can find the area under the parabola by integrating. Then add the areas under the parabolas. The result is Simpson's Rule.

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)]$$

where n is even and $\Delta x = \frac{b-a}{n}$.

Fact:

$$S_{2n} = \frac{1}{3}T_n + \frac{2}{3}M_n$$

- Use Simpson's Rule with $n = 10$ to approximate $\int_1^2 (1/x)\,dx$. Solution $S_n \approx 0.693150$.

- Error Bound for Simpson's Rule

  Suppose that $|f^{(4)}(x)| \leq K$ for $a \leq x \leq b$. If E_s is the error involved using Simpson's Rule, then

  $$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$

  It follows that Simpson's Rule gives a better approximation than the Midpoint Rule or the Trapezoid Rule.

- How large should we take n in order to guarantee that the Simpson's Rule approximation of $\int_1^2 (1/x)\,dx$ is accurate to 0.0001?

  SOLUTION $n = 8$. Remember, n must be even!

- Example.

  1. Use Simpson's Rule with $n = 10$ to approximate the integral $\int_0^1 e^{x^2}\,dx$.
  2. Estimate the error involved with this approximation.

  SOLUTION

1. $S_{10} \approx 1.462681$.
2. The fourth derivative of $f(x) = e^{x^2}$ is

$$f^{(4)}(x) = (12 + 48x^2 + 16x^4)e^{x^2}$$

This is an increasing function on [01,] and takes its maximum value at $x = 1$. Finally, we pick $K = 76e$ and the bound on the error is ≈ 0.000115.

$$\int_0^1 e^{x^2}\ dx \approx 1.463$$

7.8 Improper Integrals

- Calculate the area under the curve $f(x) = \frac{1}{x^2}$ from $x = 1$ to $x = t$. We see that the area is
$$A(t) = 1 - \frac{1}{t}$$
If we let $t \to \infty$, the approaches 1. We can say that the area under the curve on the interval $[1, \infty)$ is equal to 1.

- **Definition of Improper Integral of Type I.**

  1. If $\int_a^t f(x)\, dx$ exists for all $t \geq a$, then
  $$\int_a^\infty f(x)\, dx = \lim_{t\to\infty} \int_a^t f(x)\, dx$$
  provided this limit exists as a finite number.
  2. If $\int_t^b f(x)\, dx$ exists for all $t \leq b$, then
  $$\int_{-\infty}^b f(x)\, dx = \lim_{t\to-\infty} \int_t^b f(x)\, dx$$
  provided this limit exists as a finite number.
  The improper integrals are called divergent if the limit does not exist and convergent if the limit exists.
  3. If both $\int_a^\infty f(x)\, dx$ and $\int_{-\infty}^a f(x)\, dx$ are convergent, then we define
  $$\int_{-\infty}^\infty f(x)\, dx = \int_{-\infty}^a f(x)\, dx + \int_a^\infty f(x)\, dx$$
  Any real number a can be used for this definition.

- If f is a positive function, then the improper integral of that function can be interpreted as area under the curve.

  If $f(x) \geq 0$ and the integral $\int_a^\infty f(x)\, dx$ is convergent, then we define the are of the region under the curve f for x belonging to the interval $[a, \infty)$ to be
$$A = \int_a^\infty f(x)\, dx$$

- Example: Determine whether the integral $\int_1^{\infty}(1/x)\ dx$ is convergent or divergent. SOLUTION It is divergent.
- Example: Evaluate $\int_{-\infty}^{0} xe^x\ dx$.

  Integrate by parts.

  $$\int_{-\infty}^{0} xe^x\ dx = [xe^x - e^x]_t^0 = -te^t - 1 + e^t$$

  Take the limit, using L'Hospital's Rule. The answer is -1.
- Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\ dx$.

  SOLUTION Choose $a = 0$.

  $$\int \frac{1}{1+x^2}\ dx = \tan^{-1} x$$

  Therefore,

  $$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\ dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\ dx + \int_{0}^{\infty} \frac{1}{1+x^2}\ dx$$

  We need to calculate each separately.

  The final answer is π.
- For what value of p is the integral

  $$\int_1 \infty \frac{1}{x^p}\ dx$$

  convergent?

  We already did the case when $p = 1$ and found that it is divergent.

  Answer: $\int_1 \infty \frac{1}{x^p}\ dx$ is convergent if $p > 1$ and divergent if $p \leq 1$.
- **Type 2: Discontinuous Integrands**

  Here we deal with the integrals of functions that have numbers in their domain where the function is discontinuous. In particular we are interested in functions f such that

  $$\lim_{x \to a} f(x) = \infty.$$

  The definite integral is undefined for any interval containing a number where the function is discontinuous.

- Definition of an Improper Integral of Type 2.

  1. If f is discontinuous on $[a, b)$ and is discontinuous at b, then

  $$\int_a^b f(xf)\ dx = \lim_{t \to b^-} \int_a^t f(x)\ dx$$

     if this limit exists (as a finite number).

  2. If f is discontinuous on $(a, b]$ and is discontinuous at a, then

  $$\int_a^b f(xf)\ dx = \lim_{t \to a^+} \int_t^b f(x)\ dx$$

     if this limit exists (as a finite number).

     These improper integrals are called convergent if the limit exists, divergent if the limit does not exist.

  3. If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\ dx$ and $\int_c^b f(x)\ dx$ are convergent, then we define

  $$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$$

- Find $\int_2^5 \frac{1}{\sqrt{x-2}}\ dx$.

  SOLUTION $2\sqrt{3}$.

- Determine whether $\int_0^{\pi/2} \sec x\ dx$ converges or diverges.

  SOLUTION We need to remember that

  $$\int \sec x\ dx = \ln|\sec x + \tan x| + C$$

  The answer is that the integral diverges.

- Evaluate $\int_0^3 \frac{dx}{x-1}$ if possible.

  SOLUTION The function is undefined at $x = 1$ so it is necessary to break up the integral. The integral is divergent.

- Evaluate $\int_0^1 \ln x \; dx$.

  SOLUTION We need to integrate by parts letting $u = \ln x$ and $dv = dx$.

$$\int \ln x \; dx = x \ln(x) - x + C$$

  To calculate the limit, apply L'Hospital's Rule.

  The answer is -1.

- **Comparison Test for Improper Integrals**

  We are often interested in knowing whether an improper integral is convergent or divergent regardless of whether we can find a value for a convergent integral. This theorem holds for both Type 1 and Type 2 improper integrals.

- Comparison Theorem. Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

  1. If $\int_a^\infty f(x) \; dx$ is convergent, then $\int_a^\infty g(x) \; dx$ is convergent.
  2. If $\int_a^\infty g(x) \; dx$ is divergent, then $\int_a^\infty f(x) \; dx$ is divergent.

- Show that $\int_0^\infty e^{-x^2} \; dx$ is convergent.

  SOLUTION

  First, it is not possible to find an antiderivative of e^{-x^2} in terms of elementary functions.

  We compare with $y = e^{-x}$.

$$e^{-x^2} \leq e^{-x} \text{ for } x \geq 1$$

$$\int_0^\infty e^{-x^2} \; dx = \int_0^1 e^{-x^2} \; dx + \int_1^\infty e^{-x^2} \; dx$$

  The first integral is finite because it is an integral of a finite function on a closed finite interval.

  The second integral is convergent because $\int_0^\infty e^{-x} \; dx$ is convergent. $\square$

  Footnote: $\int_0^1 e^{-x^2} = \sqrt{\pi}/2$.

- Show that the integral $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ is divergent by the Comparison Theorem.

  SOLUTION
  $$\frac{1+e^{-x}}{x} \geq \frac{1}{x} \text{ for } x \geq 1.$$
  We showed that $\int_1^\infty (1/x)\,dx$ is divergent.

8 Further Applications of Integration

8.1 Arc Length

- We derive a formula for the length of a curve defined by a function f which has continuous first derivative on an interval $[a, b]$. Such a curve is called smooth.

  We first form a partition of n subintervals. On the curve this corresponds to points $P_0, P_1, P_2, \ldots, P_{n-1}, P_n$. We can connect these points and find a piecewise linear approximation of the curve.

  The length of each segment is given by

  $$|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$

  Here we can use the Mean Value Theorem which says there is a number x_i^*in $[x_{i-1}, x_i]$ such that

  $$\Delta y_i = f'(x_i^*)\Delta x.$$

  We therefore have

  $$|P_{i-1}P_i| = \sqrt{1 + [f'(x_i^*)]^2}\Delta x$$

  We can define arc length as

  $$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\Delta x$$

  It follows that

  $$L = \int_a^b \sqrt{1 + [f'(x)]^2}\ dx$$

- **Arc Length Formula**

  If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \le x \le b$, is

  $$L = \int_a^b \sqrt{1 + [f'(x)]^2}\ dx$$

Using Leibniz notation,

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

- Find the length of the arc of the semi-cubical parabola $y^2 = x^3$ between the points $(1, 1)$ and $(4, 8)$.

  SOLUTION For the top half of the curve, we have

  $$y = x^{3/2} \quad \frac{dy}{dx} = \frac{3}{2}x^{1/2}$$

  The arc length is therefore

  $$L = \int_1^4 \sqrt{1 + \frac{9}{4}x}\, dx$$

  Use substitution, $u = 1 + \frac{9}{4}x$. The answer is $\frac{1}{27}(80\sqrt{10} - 13\sqrt{13})$.

- If a curve is given by $x = g(y)$, $c \le y \le d$, and $g'(y)$ is continuous on $[c, d]$, then we have the following formula for arc length.

  $$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

- Find the length of the arc of the parabola $y^2 = x$ from $(0, 0)$ to $(1, 1)$.

  SOLUTION

  $$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1 + 4y^2}\, dy$$

  To solve this, make the trig substitution $y = \frac{1}{2}\tan\theta$.

  The final answer is

  $$L = \frac{\sqrt{5}}{2} + \frac{\ln(\sqrt{5} + 2)}{4} \quad \square$$

- **The Arc Length Function**

  If a smooth curve C has the equation $y = f(x)$, $a \le x \le b$, let $s(x)$ be the distance along C from the point $P_0(a, f(a))$ to the point $Q(x, f(x))$. The function s is called the arc length function and is given by

  $$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt$$

  By part 1 of the Fundamental Theorem of Calculus

  $$s'(x) = \sqrt{1 + [f'(x)]^2}$$

  The differential of arc length is

  $$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

  This equation is sometimes written as

  $$(ds)^2 = (dx)^2 + (dy)^2$$

- Find the arc length function for the curve $y = x^2 - \frac{1}{8}\ln x$ taking $P_0(1, 1)$ as the starting point.

  SOLUTION $s(x) = x^2 + \frac{1}{8}\ln x - 1.$ □

8.2 Area of a Surface of Revolution

- We find the surface area of the frustum of a cone. This is a cone with its top cut off. If the bottom radius is r_1, the top radius r_2, slant height l, then the surface area is given by

$$A = \pi(r_1 l + r_2 l)$$

i.e.

$$A = 2\pi r l, \quad r = \frac{r_1 + r_2}{2}$$

- Proof of previous formula. First, let's derive the surface area of a cone with radius r and slant height l. If we cut open the cone we get a sector of a circle with arc length l and radius r. The central angle θ equals $\frac{2\pi r}{l}$. The formula for the area of a sector is $A = \frac{1}{2}R^2\theta$. Therefore, the area of this particular sector, $R = l$ is

$$A = \frac{1}{2} l^2 \left(\frac{2\pi r}{l}\right) = \pi r l.$$

Now we look at the frustum of a cone. We will think of this as a cone with radius r_2 and slant height $l_1 + l$ with a smaller cone with radius r_1 and slant height l_1 taken away.

The surface area of the frustum is then given by

$$\begin{aligned} A &= \pi(l + l_1)r_2 - \frac{1}{2}\pi l_1 r_1 \\ &= \pi l r_2 + \frac{1}{2}\pi l_1 (r_2 - r_1) \end{aligned}$$

By similar triangles,

$$\frac{r_1}{l_1} = \frac{r_2}{l + l_1}$$

This implies that

$$r_1(l + l_1) = r_2 l_1$$

i.e.

$$r_2 l_1 - r_1 l_1 = r_1 l$$

It follows that

$$A = \pi l r_2 + \frac{1}{2}\pi r_1 l$$

i.e.

$$A = 2\pi r l, \quad r = \frac{r_1 + r_2}{2}$$

- We now derive a formula for the surface area of a solid of revolution. Suppose that a positive function f with continuous first derivative is rotated around the x-axis defined on an interval $[a, b]$.

  We make a partition $\{x_0, x_1, x_2, \ldots, x_i, \ldots, x_{n-1}, x_n\}$ and we look at the curve defined on the ith interval. If we draw line segment $P_{i-1}P_i$, we can calculate the surface area of the frustum generated by rotating line segment $P_{i-1}P_i$ about the x-axis.

  For this frustum, $r = \frac{f(x_{i-1})+f(x_1)}{2}$ and $l = |P_{i-1}P_i|$. Therefore the surface area is given by

$$A_i = 2\pi r l = 2\pi \frac{f(x_{i-1}) + f(x_1)}{2} |P_{i-1}P_i|$$

  But from last section, we saw that

$$|P_{i-1}P_i| = \sqrt{1 + [f'(x_i^*)]^2}\Delta x, \text{ some } x_i^* \in [x_{i-1}, x_i]$$

  Therefore the area of the ith frustum is

$$A_i = 2\pi \frac{f(x_{i-1}) + f(x_1)}{2} \sqrt{1 + [f'(x_i^*)]^2}\Delta x$$

  We can replace $\frac{f(x_{i-1})+f(x_1)}{2}$ by $f(x_i*)$ because as the interval gets smaller and smaller, these values become infinitely close together (f is a continuous function).

  Therefore we can write

$$A_i = 2\pi f(x_i^*)\sqrt{1 + [f'(x_i^*)]^2}\Delta x$$

  Summing over i, and letting $n \to \infty$ we define the surface area to be

$$\boxed{S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\ dx}$$

In Leibnitz notation

$$\boxed{S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx}$$

If the curve is described by $x = g(y)$, $c \le y \le d$, then the formula becomes

$$\boxed{S = \int_c^d 2\pi y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy}$$

Both formulas can be expressed symbolically as

$$\boxed{S = \int 2\pi y ds}$$

or if we rotate about the y-axis,

$$\boxed{S = \int 2\pi x ds}$$

where either

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad \text{or} \quad \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$

- The curve $y = \sqrt{4 - x^2}$, $-1 \le x \le 1$, is an arc of the circle $x^2 + y^2 = 4$. Find the area of the surface obtained by rotating this arc about the x-axis. SOLUTION 8π.
- The arc of the parabola $y = x^2$ from $(1, 1)$ to $(2, 4)$ is rotated about the y-axis. Find the area of the resulting surface. SOLUTION $\frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5})$

- Find the area of the surface generated by rotating the curve $y = e^x$, $0 \leq x \leq 1$, about the x-axis. SOLUTION

$$S = \pi[e\sqrt{1+e^2} + \ln(e + \sqrt{1+e^2}) - \sqrt{2} - \ln(\sqrt{2}+1)]$$

  Note: We must use the formula for antiderivative of $\sec^3 x$ (#71 on the table of integrals).

8.3 Applications to Physics and Engineering: Moments and Centers of Mass

- Our goal is to find the point P on which a thin plate of any given shape balances horizontally. We call this point the **center of mass** of the plate.

- Consider the one-dimensional situation where two masses m_1 and m_2 are attached to a rod of negligible mass on opposite sides of a fulcrum and distances d_1 and d_2 from the fulcrum. The rod will balance if

$$m_1 d_1 = m_2 d_2$$

This is called the Law of the Lever discovered by Archimedes.

Now suppose the masses lie along the x-axis with m_1 at x_1 and m_2 at x_2 and center of mass $\overline{x}$. Then $d_1 = x_1 = overlinex$ and $d_2 = x_2 - \overline{x}$. The equation above becomes

$$m_1(x_1 - \overline{x}) = m_2(x_2 - \overline{x})$$

$$m_1\overline{x} + m_2\overline{x} = m_1x_1 + m_2x_2$$
$$\overline{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

The numbers m_1x_1 and m_2x_2 are called the moments of the masses m_1 and m_2.

In general, if a system of n particles with masses $m_1, m_2, \ldots, m_n$ lie at points $x_1, x_2, \ldots, x_n$ on the x-axis, it can be shown that the center of mass is given by

$$\overline{x} = \frac{\sum_{i=1}^n m_ix_i}{\sum_{i=1}^n m_i} = \frac{\sum_{i=1}^n m_ix_i}{m}$$

The total mass of the system is $m = \sum_{i=1}^n m_i$ and the moment of the system about the origin is defined as $M = \sum_{i=1}^n m_ix_i$.

Therefore the above equation can by written as

$$m\overline{x} = M.$$

If the total mass were considered as begin concentrated at the center of mass $\overline{x}$, then its moment would be the same as the moment of the system.

- Consider n particles with masses $m_1, m_2, \ldots, m_n$ located at points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in the xy-plane. By analogy, we define the **moment of the system about the y-axis** to be

$$M_y = \sum_{i=1}^{n} m_i x_i$$

and the **moment of the system about the x-axis** as

$$M_x = \sum_{i=1}^{n} m_i y_i$$

Then M_y measures the tendency of the system to rotate about the y-axis and M_x measures the tendency of the system to rotate about the x-axis.

The coordinates $(\overline{x}, \overline{y})$ of the center of mass are given in terms of the moments by the formula

$$\boxed{\overline{x} = \frac{M_y}{m}, \quad \overline{y} = \frac{M_x}{m}}$$

The center of mass $(\overline{x}, \overline{y})$ is the point where a single particle of mass m would have the same moments as the system.

- Find the moments and center of mass of the system of objects that have masses 3, 4, and 8 at the points $(-1, 1), (2, -1)$, and $(3, 2)$. SOLUTION

$$\overline{x} = \frac{29}{15}, \quad \overline{y} = 1.$$

- We will no=w find the center of mass of a flat plate, called a *lamina*, with uniform density ρ that occupies a region $\mathcal{R}$ of the plane. The center of mass is called the **centroid** of $\mathcal{R}$. We will use the following principles of physics:

- The symmetry principle: If $\mathcal{R}$ is symmetric about a line l, then the centroid of $\mathcal{R}$ lies on l.

  This implies that the centroid of a rectangle is the center of the rectangle.
- If the entire mass of a region is concentrated at the center of mass, then its moments remain unchanged.
- The moments of the union of two non-overlapping regions is the sum of the moments of the individual regions.

• We first define $\mathcal{R}$ as the region below a positive, continuous function f defined on the interval $[a, b]$. We divide the interval into n subintervals. We let x_i^* be the midpoint of each sub-interval, $\overline{x}$. We construct a polygon approximation of $\mathcal{R}$.

The centroid of the ith rectangular region R_i its center given by the point $(\overline{x}_i, \frac{1}{2}f(\overline{x}_i))$.

The area of R_i is given by $f(\overline{x}_i)\Delta x$, so its mass is

$$\rho f(\overline{x}_i)\Delta x$$

The moment of R_i about the y-axis is the product of its mass and the distance C_i to the y-axis, which is x_i. Therefore

$$M_y(R_i) = [\rho f(\overline{x}_i)\Delta x]\overline{x}_i = \rho\overline{x}_i f(\overline{x}_i)\Delta x$$

Adding the moments and taking the limit as $n \to \infty$ gives

$$\boxed{M_y = \rho\int_a^b xf(x)\ dx}$$

In a similar way, we can define the moment about the y-axis.

$$\boxed{M_x = \rho\int_a^b \frac{1}{2}[f(x)]^2\ dx}$$

The mass of the plate is the density,ρ, times the area.

$$m = \rho A = \rho \int_a^b f(x)\ dx$$

The center of mass is defined so that $m\overline{x} = M_y$ and $m\overline{y} = M_x$.

Therefore,

$$\overline{x} = \frac{M_y}{m} = \frac{\rho \int_a^b xf(x)\ dx}{\rho \int_a^b f(x)\ dx} = \frac{\int_a^b xf(x)\ dx}{\int_a^b f(x)\ dx}$$

and

$$\overline{y} = \frac{M_x}{m} = \frac{\rho \int_a^b \frac{1}{2}[f(x)]^2\ dx}{\rho \int_a^b f(x)\ dx} = \frac{\int_a^b \frac{1}{2}[f(x)]^2\ dx}{\int_a^b f(x)\ dx}$$

In summary, the center of mass of the plate, or the centroid of $\mathcal{R}$, is located at the point $(\overline{x}, \overline{y})$, where

$$\boxed{\overline{x} = \frac{1}{A}\int_a^b xf(x)\ dx \qquad \overline{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\ dx}$$

- Find the center of mass of a semicircular plate of radius r.

  SOLUTION $f(x) = \sqrt{r^2 - x^2}$ and $a = -r, b = r$. By the symmetry principle, $\overline{x} = 0$.

  The area of the semicircle is $A = \pi r^2/2$, so

$$\begin{aligned}
\overline{y} &= \frac{1}{A} = \int_{-r}^{r} \frac{1}{2}[f(x)]^2\ dx \\
&= \frac{1}{\pi r^2/2}\frac{1}{2}\int_{-r}^{r} (\sqrt{r^2 - x^2})^2\ dx \\
&= \frac{2}{\pi r^2}\int_0^r (r^2 - x^2)\ dx = \frac{2}{\pi r^2}\left[r^2 x - \frac{x^3}{3}\right]_0^r \\
&= \frac{4r}{3\pi}
\end{aligned}$$

- Find the centroid of the region bounded by the curves $y = \cos x$, $y = 0$, $x = 0$, and $x = \pi/2$.

  SOLUTION   The centroid is $((\pi/2) - 1, \pi/8)$.

- If the region $\mathcal{R}$ lies between two curves $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$, then it can be shown that

$$\boxed{\overline{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]\ dx \qquad \overline{y} = \frac{1}{A}\int_a^b \frac{1}{2}\left\{[f(x)]^2 - [g(x)]^2\right\}\ dx}$$

- Find the centroid of the region bounded by the line $y = x$ and the parabola $y = x^2$. Solution $(\frac{1}{2}, \frac{2}{5})$.

- *Theorem of Pappus*   Let $\mathcal{R}$ by a plane region that lies entirely on one side of a line l in the plane. If $\mathcal{R}$ is rotated about l, then the volume of the resulting solid is the product of the area A of $\mathcal{R}$ and the distance d traveled by the centroid $\mathcal{R}$.

  Proof: Here the proof is for the special case in which the region lies between $y = f(x)$ and $y = g(x)$ and the line l is the x-axis. Using cylindrical shells, we have

$$\begin{aligned} V &= \int_a^b 2\pi x[f(x) - g(x)]\ dx \\ &= 2\pi \int_a^b x[f(x) - g(x)]\ dx \\ &= 2\pi(\overline{x}A) \\ &= (2\pi\overline{x})A = Ad \end{aligned}$$

  where $d = 2\pi\overline{x}$ is the distance traveled by the centroid during one rotation about the y-axis. $\square$.

- A torus is formed by rotating a circle of radius r about a line in the plane of the circle that is a distance $R\ \ (> r)$ from the center of the circle. Find the volume of the torus.

SOLUTION The circle has area $A = \pi r^2$. The centroid is the center and the distance traveled is $d = 2\pi R$. Therefore

$$V = Ad = (2\pi R)(\pi r^2) = 2\pi^2 r^2 R.$$

9 Differential Equations

9.3 Separable Equations

A **differential equation** is an equation that contains an unknown function and one or more of its derivatives. Here are some examples.

$$y' = xy$$

$$y'' + 2y' + y = 0$$

$$\frac{d^3y}{dx^3} + x\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = e^{-x}$$

In each of these equations, y is an unknown function of x.

The importance of differential equations lies in the fact that when a scientist or engineer formulates a physical law in mathematical terms, it frequently turns out to be a differential equation.

The **order** of a differential equation is the order of the highest derivative that occurs in the equation.

A function f is called a **solution** of a differential equation if the equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equation.

Separable Equations

A **separable equation** is a first order-differential equation that can be written in the form

$$\frac{dy}{dx} = g(x)f(y)$$

If $f(y) \neq 0$, we can write

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

where $h(y) = 1/f(y)$.

To solve we write in differential form

$$h(y)dy = g(x)dx$$

We integrate both sides:

$$\int h(y)\ dy = \int g(x)\ dx$$

To justify this procedure, use the chain rule:

$$\frac{d}{dx}\left(\int h(y)\ dy\right) = \frac{d}{dx}\left(\int g(x)\ dx\right)$$

so

$$\frac{d}{dx}\left(\int h(y)\ dy\right) = \frac{d}{dx}\left(\int g(x)\ dx\right)$$

and

$$h(y)\frac{dy}{dx} = g(x)$$

We are often interested in finding the particular solution that satisfies the condition of the the form $y(x_0) = y_0$. This is called an **initial condition**. The problem of finding a solution of the differential equation that satisfies the initial condition is called an **initial value problem**.

Example.

1. Solve the differential equation

$$\frac{dy}{dx} = \frac{x^2}{y^2}$$

   Solution $y = \sqrt[3]{x^3 + K}$.

2. Find the solution that satisfies $y(0) = 2$.

   Solution $y = \sqrt[3]{x^3 + 8}$. □

Example. Solve the differential equation

$$\frac{dy}{dx} = \frac{6x^2}{2y + \cos y}$$

SOLUTION $y^2 + \sin y = 2x^3 + C$.
Example. Solve the equation $y' = x^2 y$.
SOLUTION $y = Ae^{x^3/3}$.
Example. Solve $\dfrac{dy}{dt} = ky$.
SOLUTION $y = Ae^{kt}$.

Logistic Growth

An appropriate differential equation for modeling population growth is $y' = ky$. This says that the rate of growth is proportional to the size of the population. We say from the last example that the solution is $y = Ae^{kt}$.

In a restricted environment and with limited food supply, the population cannot exceed a maximum size M (called **carrying capacity**) at which it consumes its entire food supply.

If we make the joint assumption that the rate of growth of the population is jointly proportional to the size of the population, y, and the amount by which y falls short of the maximal size, $M - y$, then we have the equation

$$\boxed{\frac{dy}{dx} = ky(M - y)}$$

where k is constant.

This equation is called the **logistic equation** and was used by Dutch mathematician biologist Verhulst in the 1840's to model population growth.

The logistic equation is separable. We can write it in the form

$$\int \frac{dy}{y(M - y)} = \int k \; dt$$

Using partial fractions

$$\frac{1}{y(M - y)} = \frac{1}{M}\left[\frac{1}{y} + \frac{1}{M - y}\right]$$

and so we get

$$\frac{1}{M}(\ln|y| - \ln|M-y|) = kt + C$$

Because $0 < y < M$ we have $|y| = y$ and $|M - y| = M - y$. Therefore

$$\ln\frac{y}{M-y} = M(kt + C)$$

$$\frac{y}{M-y} = Ae^{kMt}$$

If the population at $t = 0$ is $y(0) = y_0$, then $A = y_0/(M - y_0)$, so

$$\frac{y}{M-y} = \frac{y_0}{M-y_0}e^{kMt}$$

Solving for y gives

$$y = \frac{y_0 M e^{kMt}}{M - y_0 + y_0 e^{kMt}} = \frac{y_0 M}{y_0 + (M - y_0)e^{-kMt}}$$

We see that

$$\lim_{t\to\infty} y(t) = M$$

9.6 Linear Equations

- A first-order **linear** differential equation is one that can be put in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where P and Q are continuous functions on a given interval.

- Note that if $Q(x) = 0$, then the equation is a first-order separable differential equation.

- Sometimes an equation has to be rearranged to be in the correct form.

$$xy' + y = 2x$$

is equivalent to

$$y' + \frac{1}{x}y = 2$$

on intervals where $y \neq 0$.

- We will now solve the equation

$$xy' + y = 2x$$

Note that

$$(xy)' = xy' + y$$

Therefore we can rewrite the equation as

$$(xy)' = 2x$$

Integrating both sides gives

$$xy = x^2 + C, \quad \text{i.e., } y = x + \frac{C}{x}$$

- All first-order linear D.E. can be solved in a similar way by multiplying the equation by what is called an *integrating factor*, $I(x)$.

We want to find $I(x)$ such that

$$I(x)(y' + P(x)y) = (I(x)y)'.$$

Suppose that we can find such a function $I(x)$. Then we would get

$$I(x)y = \int I(x)Q(x)\ dx + C$$

So the solution is

$$y(x) = \frac{1}{Ix}\left[\int I(x)Q(x)\ dx + C\right]$$

- We can derive a formula for $I(x)$.

$$I(x)(y' + P(x)y) = (I(x)y)'$$

$$I(x)y' + I(x)P(x)y = (I(x)y)' = I'(x)y + I(x)y'$$
$$I(x)P(x) = I'(x)$$

This is a separable equation for I.

$$\int \frac{dI}{I} = \int P(x)\ dx$$

$$\ln|I| = \int P(x)\ dx$$

$$I = Ae^{\int P(x)\ dx}$$

We can just let $A = 1$, because we only need one function that satisfies the requirements for I.

Therefore,

$$\boxed{I(x) = e^{\int P(x)\ dx}}$$

- *To solve the linear differential equation $y' + P(x)y = Q(x)$, multiply both sides by the integrating factor $I(x) = e^{\int P(x)\ dx}$ and integrate both sides.*

- Solve the differential equation $y' + 3x^2y = 6x^2$. SOLUTION $y = 2 + Ce^{-x^3}$.

- Find the solution to the initial-value problem

$$x^2y' + xy = 1, \quad x > 0 \;\; y(1) = 2$$

SOLUTION $y = \dfrac{\ln x + 2}{x}$

- Solve $y' + 2xy = 1$.

SOLUTION

$$y = e^{-x^2} \int e^{x^2}\ dx + Ce^{-x^2}$$

10 Parametric Equations and Polar Coordinates

10.1 Curves Defined by Parametric Equations

Given a curve C that does not necessarily represent a function, the x- and y-coordinates are sometimes given as a function of a third variable t, called the parameter.

$$x = f(t), \quad y = g(t)$$

Example. Sketch and identify the curve defined by the equations

$$x = t^2 - 2t, \quad y = t + 1$$

Solution Plot points. The result is a parabola opening to the right, shifted left by one, up by 3/2.

We can eliminate the parameter t by letting

$$t = y - 1.$$

Then we see that

$$x = (y-1)^2 - 2(y-1) = y^2 - 2y + 1 - 2y + 2 = y^2 - 4y + 3 = (y-2)^2 - 1$$

i.e.

$$(x - (-1)) = (y-2)^2$$

The values of the parameter t are sometimes restricted to an interval $[a, b]$.

For example, in the previous example, we could restrict t to

$$0 \leq t \leq 4.$$

The initial point is $(f(a), g(a))$ and the terminal point is $(f(b), g(b))$.

Example. What curve is represented by the parametric equations $x = \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$?

SOLUTION

We see that

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

So the curve is a circle. As t goes from 0 to 2π, the unit circle is traces.

Example. What curve is represented by the parametric equations $x = \cos 2t$, $y = \sin 2t$, $0 \leq t \leq 2\pi$?

SOLUTION

This is also a circle, but the point (x, y) travels twice around the circle as t goes from 0 to 2π.

Example. Sketch the curve with parametric equations $x = \sin t$ and $y = \sin^2 t$.

SOLUTION Note that $y = x^2$. So this will give a parabola, but also note that $-1 \leq x \leq 1$. So the point (x, y) will move back and forth from $(-1, l)$ to $(1, 1)$.

Example. An ellipse can be given by parametric equations

$$x(t) = a \cos t, \quad y(t) = b \sin t$$

We see that

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} &= \frac{(a \cos t)^2}{a^2} + \frac{(b \sin t)^2}{b^2} \\ &= \frac{a^2 \cos^2 t}{a^2} + \frac{b^2 \sin^2 t}{b^2} = \cos^2 t + \sin^2 t = 1 \end{aligned}$$

So the curve represented by the given parametric equations is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Example. The right branch of a hyperbola can be given by parametric equations

$$x(t) = a\cosh t, \quad y(t) = b\sinh t$$

Where

$$\cosh t = \frac{e^t + e^{-t}}{2}, \quad \sinh t = \frac{e^t - e^{-t}}{2}$$

We see that

$$\begin{aligned}\frac{x^2}{a^2} - \frac{y^2}{b^2} &= \frac{(a\cosh t)^2}{a^2} - \frac{(b\sinh t)^2}{b^2} \\ &= \frac{a^2\cosh^2 t}{a^2} - \frac{b^2\sinh^2 t}{b^2} = \cosh^2 t - \sinh^2 t = 1\end{aligned}$$

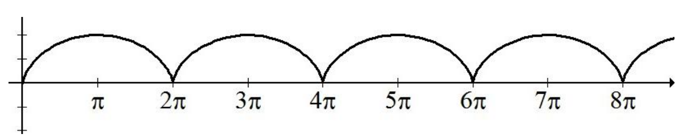
The function $x = \cosh t$ is strictly positive, so the curve represented by the given parametric equations is the right branch of the hyperbola

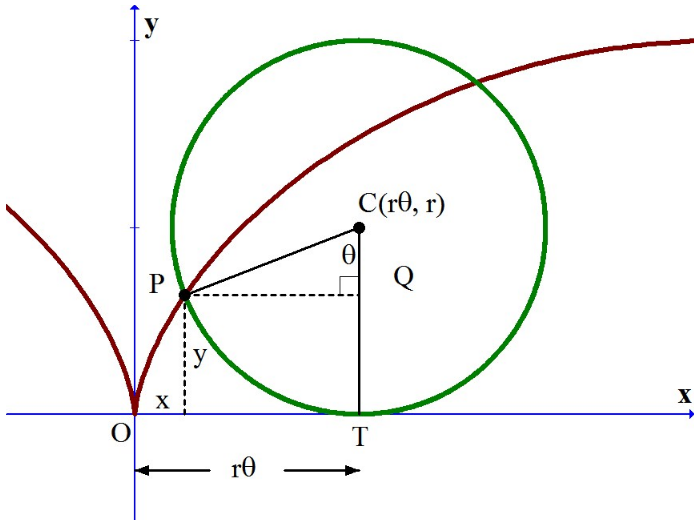
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Example. The Cycloid

The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a cycloid. If the circle has radius r and rolls along the x-axis and if one position of P is the origin, then the parametric equations are given by

$$x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta), \quad \theta \in \mathbb{R}$$




y
C(rθ, r)
θ
P
Q
y
x
x
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T
rθ

10.2 Calculus with Parametric Curves

Tangents

Suppose that a curve given by parametric equations $x = f(t)$ and $y = g(t)$ can be expressed in the form $y = F(x)$. Then substituting for x and y we get

$$g(t) = F(f(t)).$$

If we differentiate with respect to t, using the chain rule, we get

$$g'(t) = F'(f(t))f'(t) = F'(x)f'(t)$$

If $f'(t) \neq 0$, then

$$F'(x) = \frac{g'(t)}{f'(t)}$$

The slope of the tangent line to the curve is therefore given by

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}, \quad \text{if } \frac{dx}{dt} \neq 0$$

It can be shown the tangent line is horizontal when $dy/dt = 0$ and vertical when $dx/dt = 0$, provided that they are not both zero at the same time.

We can find d^2y/dx^2 by replacing y by dy/dx in the previous formula.

$$d^2y/dx^2 = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Example. A curve C is defined by the parametric equations $x = t^2$, $y = t^3 - 3t$.

1. Show that C has two tangents at the point $(3, 0)$ and find their equations.

   SOLUTION $y = \pm\sqrt{3}(x - 3)$.

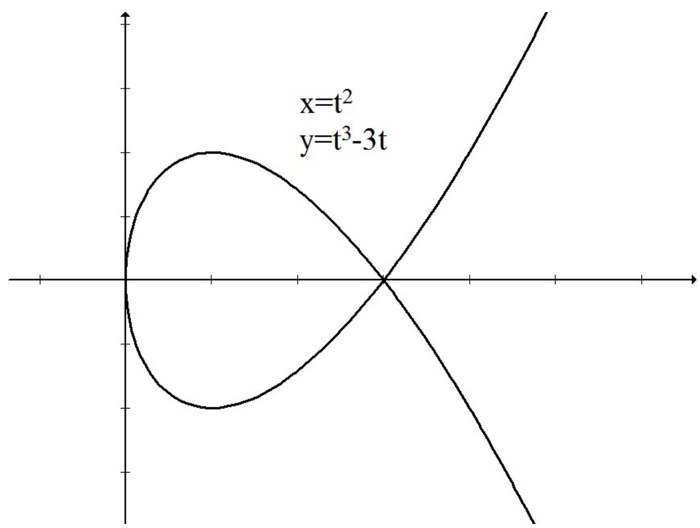
2. Find the points on C where the tangent is horizontal or vertical.

   SOLUTION Horizontal at $(1, -2)$ and $(1, 2)$, vertical at $(0, 0)$.

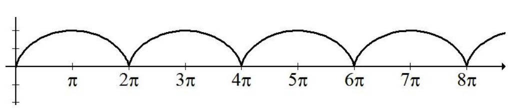
3. Determine where the curve is concave upward or downward.

   SOLUTION Concave up when $t > 0$, down when $t < 0$.

4. Sketch the curve.



Example. The cycloid is the path of a piece of gum stuck on the a wheel.



Find the tangent to the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$ at the point where $\theta = \pi/3$.

SOLUTION

$$y - r/2 = \sqrt{3}\left(x - r\pi/3 + r\sqrt{3}/2\right)$$

At what points is the tangent line horizontal? When is it vertical?

SOLUTION $((2n-1)\pi r, 2r)$.

Areas

If a curve given by $y = F(x)$, $a \le x \le b$, is a positive function, can be described by parametric equations $x = f(t)$ and $y = g(t)$ and is traversed once as t increased from α to β, then we can find the area under the curve by using the substitution rule.

$$A = \int_a^b y\ dx = \int_\alpha^\beta g(t)f'(t)\ dt$$

or, if $(f(\beta), g(\beta))$ is the left endpoint,

$$A = \int_\beta^\alpha g(t)f'(t)\ dt$$

Find the area under one arch of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.

SOLUTION One arc of the cycloid is given by $0 \le \theta \le 2\pi$. Use the substitution rule to get

$$A = \int_0^{2\pi r} y\ dx = \int_0^{2\pi} r(1-\cos\theta)r(1-\cos\theta)\ d\theta = 3\pi r^2$$

Arc Length

If a curve C is given in the form $y = F(x)$, $a \le x \le b$, and F' is continuous, then

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

Suppose that C can be described by parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \le t \le \beta$, where $dx/dt = f'(t) > 0$. This means C is traversed once, from left to right, as t increases from α to β and $f(\alpha) = a$, $f(\beta) = b$. Using the substitution rule

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\ \frac{dx}{dt}\ dt$$

because $dx/dt > o$, we have

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

In fact, we can use this formula for cases where the curve does not represent a function. We can derive the formula directly using Reimann sums.

This is consistent with the formula $L = \int ds$ and $ds^2 = dx^2 + dy^2$.

Example. If we use the representation of the unit circle,

$$x = \cos t, \quad y = \sin t, \quad 0 \leq t \leq 2\pi,$$

use the formula to find the arc length.

Example. If we use

$$x = \sin 2t, \quad y = \cos 2t, \quad 0 \leq t \leq 2\pi,$$

then we get an answer of 4π because the circle is traveled twice.

Example. Find the length of one arch of the cycloid

$$x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta).$$

SOLUTION $0 \leq \theta \leq 2\pi$, the answer is 8π.

Surface Area

We can derive a formula for surface. If the curve given by $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, is rotated about the x-axis, where f', g' are continuous and $g(t) \geq 0$, then the area of the resulting surface is given by

$$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

The general form is $S = \int 2\pi y\ dx$ and $S = \int 2\pi x\ ds$ but we can use

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example. Show that the surface area of a sphere of radius r is $4\pi r^2$.

SOLUTION The sphere is obtained by rotating the semicircle

$$x = r\cos t, \quad y = r\sin t, \quad 0 \le t \le \pi$$

The answer is $4\pi r^2$.

10.3 Polar Coordinates

Example. Plot the points given in polar coordinates (r, θ).

1. $(1, \pi/6)$
2. $(4, -\pi/4)$
3. $(-1, 3\pi/4)$

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

$$x^2 + y^2 = r^2$$
$$\frac{y}{x} = \tan\theta$$

Example. Convert from polar coordinates (r, θ) to rectangular coordinates (x, y).

1. $(1, \pi/3)$
2. $(4, 5\pi/6)$

Example. Convert from rectangular coordinates (x, y) to polar coordinates (r, θ).

1. $(4, 4)$
2. $(6, -3)$
3. $(-1, 1)$
4. $(-2, -3)$

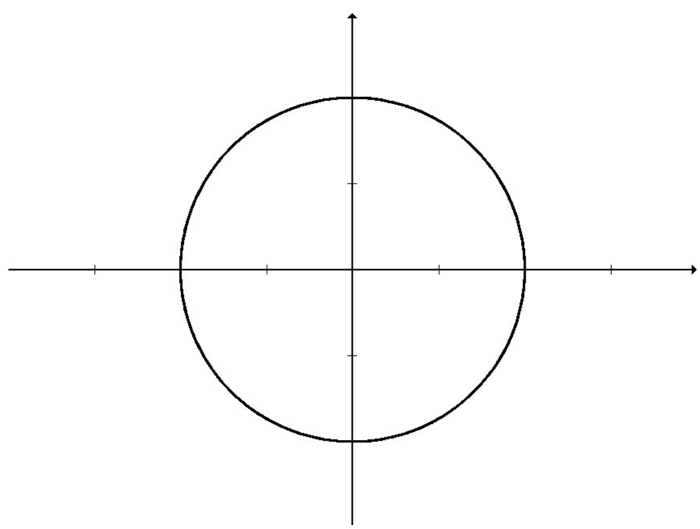
The Graphs of Curves in Polar Coordinates

A polar curve is given by a formula $r = f(\theta)$.

Example. Sketch the graph of the polar curve.

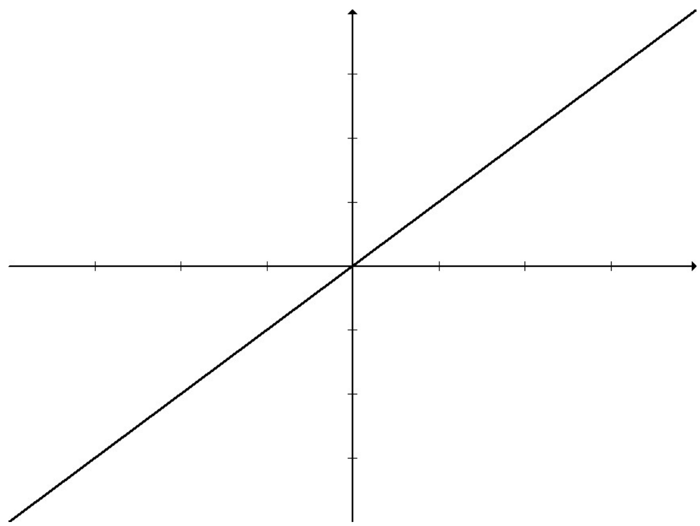
1. The equation of circle of radius $r = 2$ centered at the origin.

$$r = 2$$



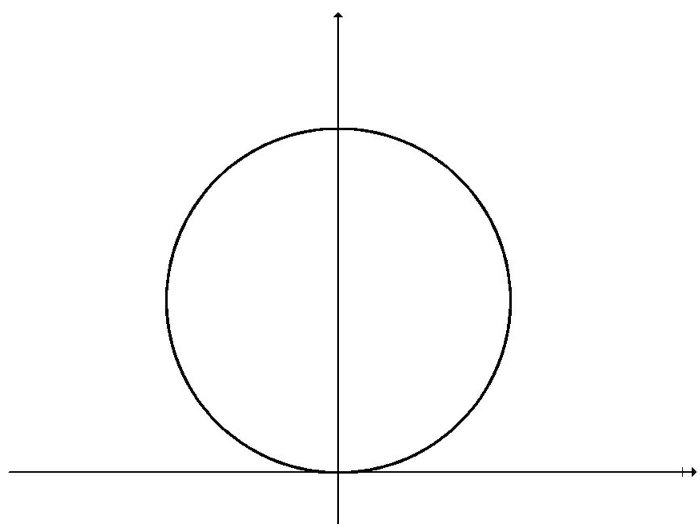
2. A line through the origin forming angle $\theta = \pi/4$ with x-axis.

$$\theta = \pi/4$$



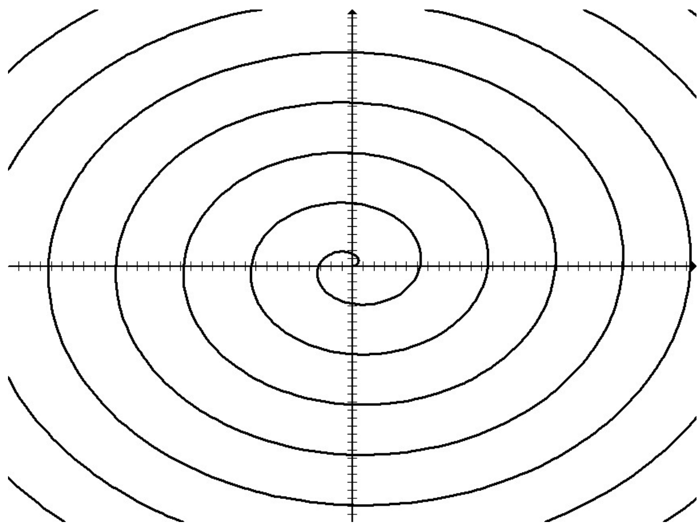
3. A circle.

$$r = \sin\theta$$



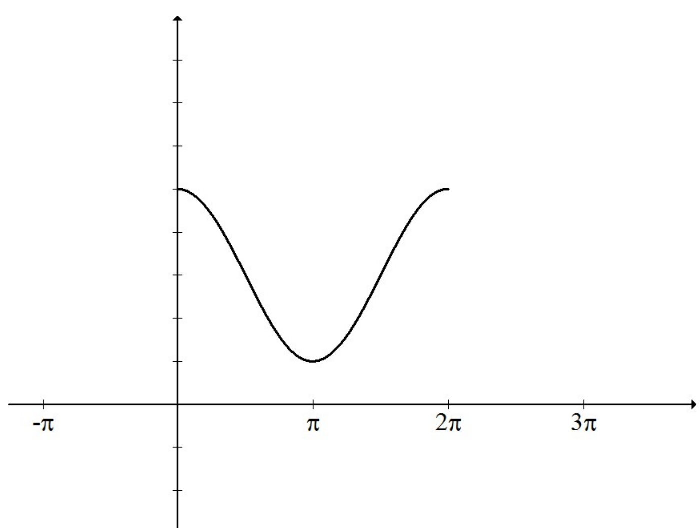
4. Archimedian Spiral

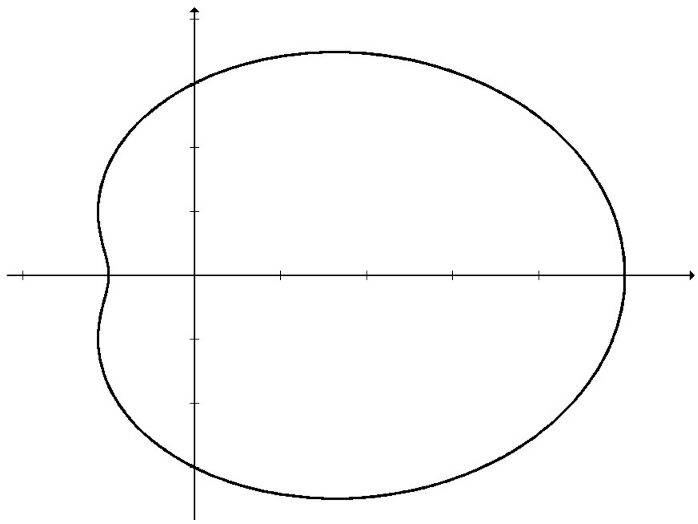
$$r = \theta, \quad \theta \geq 0$$



5. A Limacon

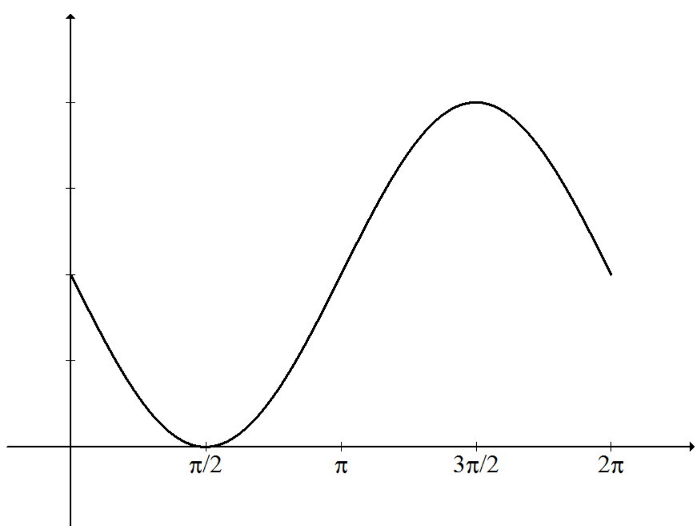
$$r = 3 + 2\cos\theta$$

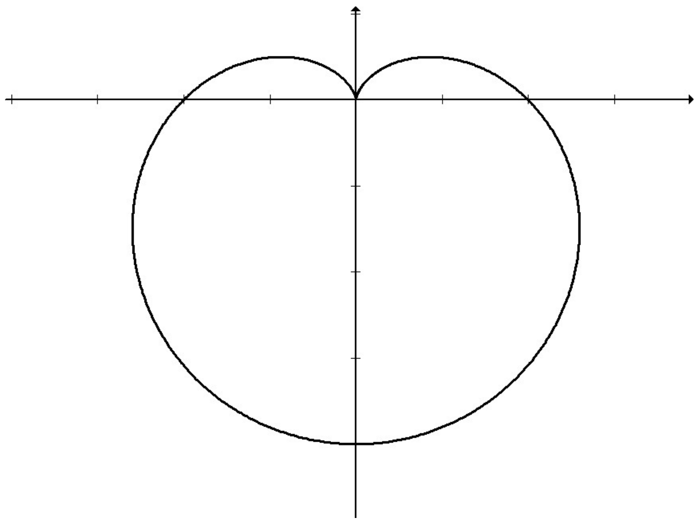




6. A Cardoid

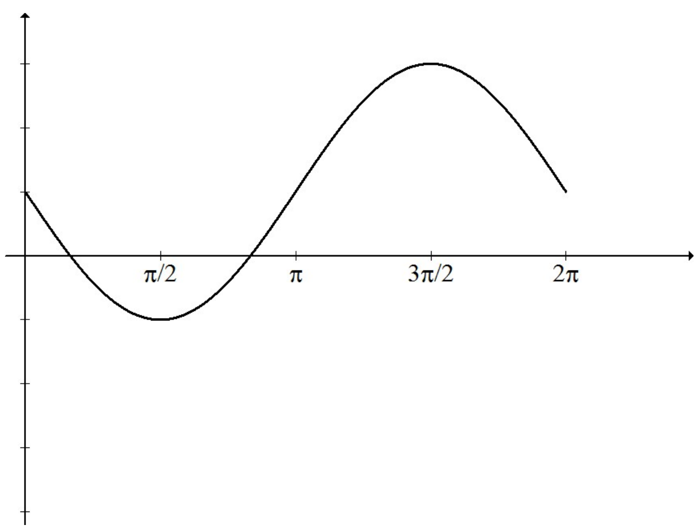
$$r = 2 - 2\sin\theta$$

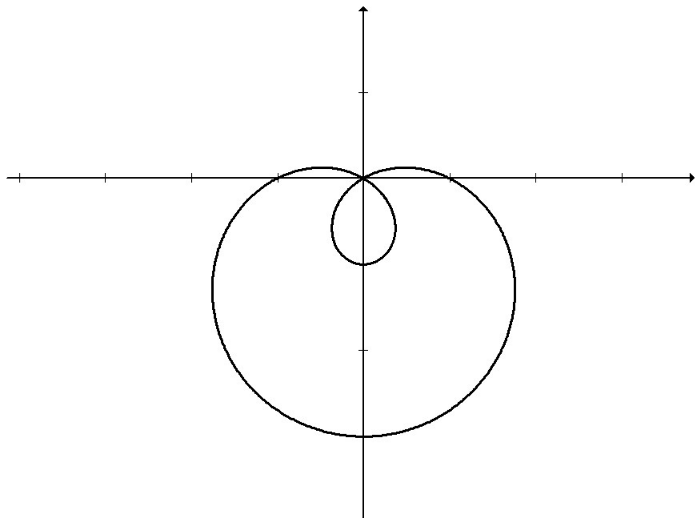




7. A Limacon with a Loop

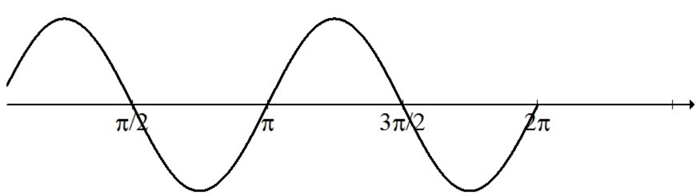
$$r = 1 - 2\sin\theta$$

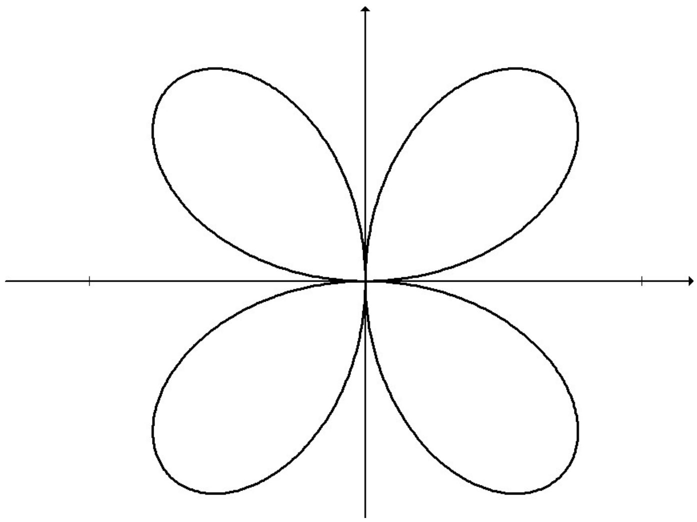




8. A Four-Leaf Rose

$$r = \sin 2\theta$$





10.4 Calculus in Polar Coordinates

$$\begin{aligned} x &= r\cos\theta = f(\theta)\cos\theta \\ y &= r\sin\theta = f(\theta)\sin\theta \end{aligned}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$\left.\frac{dy}{dx}\right|_{\theta=a} = \frac{f'(a)\sin a + f(a)\cos a}{f'(a)\cos a - f(a)\sin a}$$

Example. Find the slope of the tangent line to the three leaf rose $r = \sin 3\theta$ at $\theta = 0$ and $\theta = \frac{\pi}{4}$.
SOLUTION $0, 1/2$

Area Bounded by a Polar Curve $r = f(\theta)$

The area of the sector with central angle θ is $A = \frac{1}{2}r^2\theta$. If we partition the region bounded by a polar curve $r = f(\theta)$. Each piece is a sector with area $\frac{1}{2}(f(\theta))^2\,d\theta$. We sum the sectors to get the formula

$$A = \int_a^b \frac{1}{2} \left(f(\theta)\right)^2 \; d\theta$$

Example. Find the area of one leaf of the rose $r = \sin 3\theta$.

SOLUTION $\pi/12$

Example. Find the area of the inner loop of the limacon $r = 2 - 3\sin\theta$.
SOLUTION ≈ 0.44

Example. Find the area inside the limacon $r = 3 + 2\cos\theta$ and outside the circle $r = 2$.

SOLUTION $\dfrac{33\sqrt{3} + 44\pi}{6} - \dfrac{8\pi}{3} = \dfrac{33\sqrt{3} + 28\pi}{6}$

10.5 Conic Sections

- Definition: A conic section is the intersection of a plane and a double-napped cone. The four basic conics are formed when the plane does not pass through the vertex: circle, ellipse, parabola, and hyperbola. If the plane passes through the vertex, a degenerate conic is formed: a point, a line, or two intersecting lines.

- Definition: a parabola is the set of all points (x, y) in a plane that are equidistant from a fixed line, the directrix, and a fixed point, the focus, not on the line. The vertex is the midpoint between the focus and the directrix. The axis of the parabola is the line passing through the focus and the vertex.

- Standard Equation of a Parabola (Vertex at Origin)

  The standard equation of a parabola with vertex $(0, 0)$ and directrix $y = -p$ is
  $$x^2 = 4py, \; p \neq 0.$$
  for directrix $x = -p$, the equation is
  $$y^2 = 4px, \quad p \neq 0.$$
  The focus is on the axis p units (directed distance from the vertex.

- Example: Find the focus of a parabola whose equation is $y = -2x^2$.

  SOLUTION
  $$x^2 = 4py$$
  $$y = -2x^2 \Rightarrow x^2 = -\frac{1}{2}y = 4\left(-\frac{1}{8}\right)y.$$
  Therefore $p = -\frac{1}{8}$.

- Ellipses: An ellipse is the set of all points (x, y) in a plane the sum of whose distances from two distinct points (foci) is constant.

- The line through the foci intersect the ellipse at two points (vertices). The chord joining the vertices is the major axis, and its midpoint is the center of the ellipse. The chord perpendicular to the major axis at the center is the minor axis.

- Standard Equation of an Ellipse centered at the origin with major and minor axes of lengths $2a$ and $2b$, $0 < b < a$ is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

  or

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$

  The vertices and foci lie on the major axis, a and c units, respectively, form the center. The numbers a, b, and c satisfy

$$c^2 = a^2 - b^2$$

- Sketch an Ellipse: Sketch the ellipse given by

$$4x^2 + y^2 = 36$$

  First, rewrite as (is this necessary?)

$$\frac{x^2}{9} + \frac{y^2}{36} = 1$$

- Circles: A circle is the set of all points (x, y) in the plane with constant distance r from a point, (h, k), the center.

- Standard Equation of a Circle Centered at the Origin.

$$x^2 + y^2 = r^2$$

  If the center is (h, k), the standard equation is

$$(x - h)^2 + (y - k)^2 = r^2$$

- Write the standard form of the equation of a circle with center $(2, -1)$ and radius $r = 3$. Sketch the circle.
- Find the center and radius of the circle. Sketch the graph.

$$(x-2)^2 + (y+1)^2 = 25$$

- Hyperbolas: A hyperbola is the set of all points (x, y) in plane the difference of whose distances form two distinct points (foci) is a positive constant.
- The graph has two disconnected parts (branches). The line through the two foci intersect the hyperbola at two points (vertices). The line segment connecting the vertices is the transverse axis, and the midpoint of the transverse axis is the center of the hyperbola.
- Standard Equation of a Hyperbola (Centered at the Origin):

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{Transverse axis horizontal}$$

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad \text{Transverse axis vertical}$$

  The vertices and foci are, respectively, a and c units form the center. Moreover, a, b, and c are related by $b^2 = c^2 - a^2$.

- The asymptotes of a hyperbola: Each hyperbola has two asymptotes that intersect the center. They pass through the vertices of a rectangle with dimensions $2a$ by $2b$.

  The conjugate axis is the line segment joining $(0, b)$ and $(0, -b)$ (or $(-b, 0)$ and $(b, 0)$).

  The Asymptotes are given by

$$y = \pm\frac{b}{a}x \text{ or } y = \pm\frac{a}{b}x.$$

  The key is to draw the box.

- Write the hyperbola in standard form. Sketch the hyperbola, labeling intercepts and drawing asymptotes.

$$4x^2 - y^2 = 16$$

11 Infinite Sequences and Series

11.1 Sequences

- Definition: A sequence is a function whose domain is the set of positive integers.

  A sequence can be thought of a as a listing of numbers in a definite order:

$$a_1, a_2, \ldots, a_n$$

- Notation: The sequence $\{a_1, a_2, a_3, \ldots\}$ can be denoted $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$.

- Example: We can write a sequence using any of the following.

$$\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} \qquad a_n = \frac{n}{n+1} \qquad \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots\right\}$$

- Find the formula for the general term a_n of the sequence

$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{25}, -\frac{6}{625}, \frac{7}{3125}\right\}$$

  SOLUTION $\quad a_n = (-1)^{n-1}\dfrac{n+2}{5^n}$

- Some sequences do not have simple defining sequences.

  Let $\{p_n\}$ be the sequence where p_n is the world population as of January 1 in the year n.

- Some sequences are defined recursively. The Fibonacci sequence $\{f_n\}$ defined by

$$f_1 = 1 \quad f_2 = 1 \quad f_n = f_{n-1} + f_{n-2}, \quad n \geq 3$$

- The sequence $a_n = n/(n+1)$ can be plotted as a graph in the plane of discrete points. We can see that this sequence has a limit of 14.

- Definition: A sequence $\{a_n\}$ has a limit L an we write

$$\lim_{n\to\infty} a_n = L$$

  if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence converges (or is convergent). Otherwise, we say the sequence diverges (or is divergent).

- Definition: A sequence $\{a_n\}$ has a limit L an we write

$$\lim_{n\to\infty} a_n = L$$

  if for every $\epsilon > 0$ there is a corresponding integer N such that

$$|a_n - L| < \epsilon \text{ whenever } n > N$$

  The only difference between the definition of the limit of a sequence and a real valued function as $x \to \infty$ is that for a sequence, we use integers rather than real numbers. This leads us to the following theorem.

- Theorem If $\lim_{n\to\infty} f(x) = L$ and $f(n) = a_n$, where n is an integer, then $\lim_{n\to\infty} a_n = L$.

- Example: We know that $\lim_{x\to\infty}(1/x^r) = 0$ when $r > 0$. Therefore

$$\lim_{n\to\infty} \frac{1}{n^r} = 0 \text{ if } r > 0$$

- Definition $\lim_{n\to\infty} a_n = \infty$ means that for every positive integer M there is an integer N such that

$$a_n > M \text{ whenever } n > N$$

- The Limit Laws from Section 2.3 also hold for the limits of sequences. If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

  1. $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$

2. $\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$
3. $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$
4. $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
5. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$.
6. $\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p$ if $p > 0$ and $a_n > 0$

- The Squeeze Theorem:

  If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

- Theorem: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

  The proof is exercise 69.

- Example: Find $\lim_{n\to\infty} \frac{n}{n+1}$.

  SOLUTION Divide the numerator and denominator by n.

- Example: Calculate $\lim_{n\to\infty} \frac{\ln n}{n}$.

  We can find the the limit by using L'Hospital's Rule of the function $(\ln x)/(x)$.

- Example: Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent.

- Evaluate $\lim_{n\to\infty} \frac{(-1)^n}{n}$ if it exists.

  SOLUTION The absolute value of the sequence is 0, therefore the sequence has limit 0.

- Example: Discuss the convergence of the sequence $a_n = n!/n^n$.

  SOLUTION Use the Squeeze Theorem. The answer is 0.

- The sequence $\{r^n\}$ is convergent if $-1 < r \leq\leq 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if} \quad -1 < r < 1 \\ 1 & \text{if} \quad r = 1 \end{cases}$$

- Definition: A sequence $\{a_n\}$ is called increasing if $a_n < a_{n+1}$ for all $n \geq 1$. It is called decreasing if $a_n > a_{n+1}$ for all $n \geq 1$. It is called monotonic if it is either increasing or decreasing.

- Show that the sequence $a_n = \frac{n}{n^2+1}$ is decreasing.

- Definition A sequence $\{a_n\}$ is called bounded above if there is a number M if there is a number M such that

$$a_n \leq M \text{ for all } n \geq 1$$

  It is bounded below if there is a number m such that

$$m \leq a_n foralln \geq 1$$

  If it is bounded above and below, then $\{a_n\}$ is a bounded sequence.

- Monotonic Sequence Theorem Every bounded, monotonic sequence is convergent.

- Example Investigate the sequence defined by the recurrence relation

$$a_1 = 2 \quad a_{n+1} = \frac{1}{2}(a_n + 6) \quad \text{for } n = 1, 2, 3, \ldots$$

  SOLUTION Show that the sequence is increasing using mathematical induction. Then show that the sequence is bounded above by 6 using induction. Therefore it is convergent by the Monotonic Sequence Theorem.

  Once we know that the limit exists, we can use the recurrence relation to find the limit.

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2}(a_n + 6) = \frac{1}{2}(L + 6)$$

  Solving the equation for L gives $L = 6$.

11.2 Series

- If we add the terms in an infinite sequence, we get what is called an infinite series denoted by

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad \sum a_n$$

- Does it make sense to talk about an infinite sum?

  Let's look at the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} \ldots$$

  We define partial sums as follows:

$$\begin{aligned} s_1 &= a_1 = \frac{1}{2} \\ s_2 &= a_1 + a_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \\ s_3 &= a_1 + a_2 + a_3 = s_2 + a_3 = \frac{3}{4} + \frac{1}{8} = \frac{7}{8} \\ s_4 &= = a_1 + a_2 + a_3 + a_4 = s_3 + a_4 = \frac{7}{8} + \frac{1}{16} = \frac{15}{16} \end{aligned}$$

  Apparently $s_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}$. We see that this expression has limit of 1. Therefore we say that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

- If the sequences $s_n = a_1 + a_2 + a_3 + \ldots + a_n$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum_{n=1}^{\infty} a_n$ is called convergent and we write

$$\sum_{n=1}^{\infty} a_n = s$$

  The number s is called the sum of the series. Otherwise, the series is called divergent.

- The Geometric Series:

$$a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} + \ldots = \sum_{n=1}^{\infty} ar^{n-1}, \quad a \neq 0$$

There is a formula for this infinite sum if $-1 < r < 1$.

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

This formula is found by adding the terms s_n and rs_n, solving for s_n, and then taking the limit.

> The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ is convergent if $|r| < 1$, and its sum is
>
> $$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$
>
> If $|r| \geq 1$, the geometric series is divergent.

- Example: Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \ldots$$

SOLUTION 3

- Is the series $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$ convergent or divergent? SOLUTION Divergent.

- Write the number $2.3\overline{17}$ as a ration of integers. SOLUTION $\frac{1147}{495}$.

- Find the sum of the series $\sum_{n=0}^{\infty} x^n$, where $|x| < 1$. SOLUTION $\frac{1}{1-x}$.

- Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and find its sum.

SOLUTION

$$\begin{aligned} s_n &= \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{1+i} \right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

The answer is 1.

- Show that the harmonic series

$$\sum_{n=1}^{\infty} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

is divergent.

SOLUTION

$$\begin{aligned} s_1 &= 1 \\ s_2 &= 1 + \frac{1}{2} \\ s_4 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2} \\ s_8 &> 1 + \frac{3}{2} \\ s_{16} &> 1 + \frac{4}{2} \end{aligned}$$

Apparently,

$$s_{2^n} > 1 + \frac{n}{2}$$

Therefore the series is divergent.

- Theorem If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Proof

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$$

Note that $\lim_{n\to\infty} \frac{1}{n} = 0$ but the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

- The Test for Divergence If $\displaystyle\lim_{n\to\infty} a_n$ does not exist or if $\displaystyle\lim_{n\to\infty} a_n \neq 0$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

- Theorem If $\sum a_n$ and $\sum b_n$ are convergent, then so are the series $\sum ca_n$, $\sum(a_n + b_n)$, and $\sum(a_n - b_n)$, and

  1. $\sum ca_n = c \sum a_n$
  2. $\sum(a_n + b_n) = \sum a_n + \sum b_n$
  3. $\sum(a_n - b_n) = \sum a_n - \sum b_n$

  These properties follow from the Limit Laws when applied to the partial sums of the series.

- Find the sum of the series

$$\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n} \right)$$

  SOLUTION The first sum was found in a prior example; the second sum is a geometric series. The answer is 4.

- Note: A finite number of terms does not affect the convergence or divergence of a series.

11.3 The Integral Test and Estimate of Sums

- **The Integral Test** Suppose that f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_0^{\infty} f(x)\, dx$ is convergent. In other words:

  1. If $\int_0^{\infty} f(x)\, dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
  2. If $\int_0^{\infty} f(x)\, dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

- **Example:** Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ from convergence or divergence. SOLUTION Convergent.

- **Example:** For what values of p is the series $\int_{n=1}^{\infty} \frac{1}{n^p}$ is convergent? SOLUTION Convergent if $p > 1$ and divergent if $p \leq 1$.

- *The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.*

- **Example:**

  1. The series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent because it is a p-series with $p = 3 > 1$.
  2. The series
     $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ is divergent because it is a p-series with $p = \frac{1}{3} < 1$.

- **Remainder Estimate for the Integral Test** Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)\, dx \leq R_n \leq \int_n^{\infty} f(x)\, dx$$

- **Example:**

  1. Approximate the sum of the series $\sum 1/n^3$ by using the sum of the first 10 terms. Estimate the error involved in this approximation. SOLUTION ≈ 1.1975, $R_{10} < \frac{1}{200}$
  2. How many terms are required to ensure that the sum is accurate within 0.0005? SOLUTION We need 32 terms.

- Add s_n to the inequality in the Remainder Estimate and we get

$$s_n + \int_{n+1}^{\infty} f(x)\ dx \leq s \leq s_n + \int_{n}^{\infty} f(x)\ dx$$

- Use the inequality above with $n = 10$ to estimate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

  SOLUTION $1.201664 \leq s \leq 1.202532$.

- **Proof of Integral Test:** Optional.

11.4 The Comparison Tests

- **The Comparison Test:** Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms.

  1. If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.
  2. If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

- Example: Determine wither the series

$$\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$$

  converges or diverges. SOLUTION It is convergent by the Comparison Test.

- Example: Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ for convergence or divergence. SOLUTION Divergent by the Comparison Test.

- **The Limit Comparison Test:** Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

  where c is a finite number and $c > 0$, then either both the series converge or both diverge.

- Example: Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergence. SOLUTION Convergent by the Limit Comparison Test.

- Example: Determine whether the series

$$\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$$

  converges or diverges.

- **Estimating Sums**

  It is possible to estimate the remainder of a convergent series $\sum a_n$ by comparing it to the remainder of another convergent series, $\sum b_n$.

  Let
  $$R_n = s - s_n = a_{n+1} + a_{n+2} + \dots$$
  and
  $$T_n = t - t_n = b_{n+1} + b_{n+2} + \dots$$
  If $a_n \le b_n$ for all n, we have $R_n \le T_n$.

- Example: Use the sum of the first 100 terms to approximate the sum of the series $\sum 1/(n^3 + 1)$. Estimate the error involved in this approximation.

  SOLUTION: The error is less than 0.005 and, using a computer, the sum $\sum_{n=1}^{1} 001/(n^3 + 1) \approx 0.6864538$.

11.5 Alternating Series

- An alternating series is a series whose terms are alternatingly positive and negative. For example,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^{n}}{n+1}$$

- **The Alternating Series Test:** If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 + \ldots \quad (b_n > 0)$$

satisfies

1. $b_{n+1} \leq b_n$ for all n
2. $\lim_{n\to\infty} b_n = 0$ then the series is convergent.

- **Example:** The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

satisfies the criteria of the Alternating Series Test and therefore is convergent.

- **Example:** The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n} 3n}{4n-1}$$

is alternating but

$$\lim_{n\to\infty} b_n = \frac{3}{4}$$

Therefore we cannot apply the Alternating Series Test.

We see that the following limit does not exist, and therefore by the Test for Divergence, the series is divergent.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^{n} 3n}{4n-1}$$

- **Example:** Test the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$$

for convergence or divergence.

SOLUTION:

1. To show that the terms $b_n = n^2/(n^3+1)$ are decreasing, look at the the function

$$f(x) = \frac{x^2}{x^3+1}$$

We see that the first derivative is negative for $x \geq \sqrt[3]{2}$, and therefore b_n is decreasing for $n \geq 2$.

2. Also

$$\lim_{n\to\infty} b_n = \lim_{n\to} \frac{n^2}{n^3+1} = 0$$

Therefore the series is convergent by the Alternating Series Test.

- **Alternating Series Estimate Theorem** If $s = \sum(-1)^{n-1}b_n$ is the sum of an alternating series that satisfies

1. $b_{n+1} \leq b_n$ for all n
2. $\lim_{n\to\infty} b_n = 0$

then

$$|R_n| = |s - s_n| \leq b_{n+1}$$

- **Example:** Find the sum of the series

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$$

correct to three decimal places.

SOLUTION

– Firs show that the series converges by the Alternating Series Test.

- Write out the terms of the series. We see that

$$b_7 = \frac{1}{5040} \leq \frac{1}{5000} = .0002$$

- $s_6 \approx 0.368056$
- The error of less than 0.0002 does not affect the third decimal place, so we have

$$s \approx 0.368$$

correct to three decimal places.

11.6 Absolute Convergence and the Ratio and Root Tests

- **Definition** A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.
- **Example** The series
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$$
is absolutely convergent because the absolute value of its terms form a p-series with $p = 2$.
- **Example** The alternating series given below is convergent but not absolutely convergent.
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$
It passes the Alternating Series Test, but the absolute value of its terms for the harmonic series which is divergent.
- **Definition** A series $\sum a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.
- **Theorem** If a series $\sum a_n$ is absolutely convergent, then it is convergent.
- **Example** Determine whether the following series is convergent or divergent.
$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$$
SOLUTION The term is not an alternating series, although it has both positive and negative terms. We can apply the Comparison Test to the absolute value of its terms.
$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$$

We have

$$\frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$$

The series $\sum 1/n^2$ is a convergent p-series, and therefore by the Comparison Test, the series $\sum |\cos n|/n^2$ is convergent. Therefore, the original series is convergent.

- **The Ratio Test**

  1. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
  2. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$ then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
  3. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then the Ratio Test is inconclusive.

- **Example** Test the series $\sum_{n\to\infty} (-1)^n \frac{n^3}{3^n}$ for absolute convergence.

  SOLUTION The series passes the Ration Test, and so it is absolutely convergent.

- **Example** Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

  SOLUTION Using the Ration Test shows that

$$\frac{a_{n+1}}{a_n} \to e > 1$$

  Therefore the series is not absolutely convergent. Because all of the terms are positive, the series is divergent.

- **The Root Test**

  1. If $\lim_{n\to\infty} \sqrt[n]{|a_n| =}L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
  2. If $\lim_{n\to\infty} \sqrt[n]{|a_n| =}L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n| =}L = \infty$ then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

3. If $\lim_{n\to\infty} \sqrt[n]{|a_n| =}L = 1$, then the Ratio Test is inconclusive.

- **Example** Test the convergence of the series

$$\sum_{n\to\infty}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$$

SOLUTION The series converges by the Root Test.

- **Rearrangements**

If a series is absolutely convergent, then any rearrangement of terms will result in the same sum. If a series is conditionally convergent then Reimann proved that for any real number r, there is a rearrangement of the series that has sum r.

11.7 Strategy for Testing Series

- 1. If the series is of the form $\sum 1/n^p$, it is a p-series, which is convergent for $p > 1$ and divergent for $p \leq 1$.

  2. If the series has the form $\sum ar^{n-1}$, it is a geometric series, which converges for $|r| < 1$ and diverges for $r \geq 1$.

  3. If the series has a form similar to a p-series or geometric series, then one of the comparison tests should be considered. If the series has negative terms, we can test for absolute convergence using one of the comparison tests.

  4. If you see that $\lim_{n\to\infty} a_n \neq 0$, then the Test for Divergence should be used.

  5. If the series is of the form $\sum(-1)^{n-1}b_n$ or $\sum(-1)^n b_n$, then the Alternating Series Test is a possibility.

  6. Series that involve factorials or other products are often tested using the Ration Test. Note that $|a_{n+1}/a_n| \to 1$ as $n \to \infty$ for all p-series and therefore all rational or algebraic functions of n. Thus, the Ratio Test should not be used for such series.

  7. If a_n is of the form $(b_n)^n$, then the Root Test may be useful.

  8. If $a_n = f(n)$, where $\int_1^\infty f(x)\ dx$ is easily calculated, then the Integral Test is effective (assuming that both hypotheses of this test are satisfied).

- **Example** $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$

  SOLUTION Because $a_n \to 1/2 \neq 0$, the series diverges by the Test for Divergence.

- **Example** $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$

  SOLUTION Because a_n is an algebraic function of n, use the Limit Comparison Test with the p-series given by $b_n = \sqrt{n^3}/3n^3 = 1/3n^{3/2}$.

- **Example** $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

SOLUTION It is easy to evaluate

$$\int_1^\infty xe^{-x^2}\ dx,$$

so therefore use the Integral Test. The Ratio Test also works.

- **Example** $\int \sum_{n=1}^{\infty}(-1)^n \frac{n^3}{n4+1}$

SOLUTION Since the series is alternating, we use the Alternating Series Test.

- **Example** $\int_{k=1}^{\infty} \frac{2^k}{k!}$

SOLUTION Since the series involves $k!$, we use the Ratio Test.

- **Example** $\sum_{n=1}^{\infty} \frac{1}{2+3^n}$

SOLUTION Use the Comparison Test against the series $\sum 1/3^n$.

11.8 Power Series

- **Definition:** A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

- For each fixed x, the series becomes a series of constants that either converges or diverges.

- Power Series can be thought of as infinite polynomials.

- A **power series in** $(x-a)$ or a **power series centered at** a or a **power series about** a is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \dots$$

-

- **Example:** For what values of x is the series $\sum_{n=0}^{\infty} n! x^n$ convergent?

  SOLUTION The series is divergent by the ratio test. The series converges only when $x = 0$

- **Example:** For what values of x is the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

  SOLUTION $[2, 4)$

- **Example:** Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

  SOLUTION Dom $J_0 = \mathbb{R}$

- **Theorem:** For a given power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are only three possibilities:

  1. The series converges only when $x = a$.

2. The series converges for all x.
3. There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

- The number R is called the radius of convergence of the power series. In case (1), we say $R = 0$, and in case (2), the radius of convergence we say $R = \infty$.

- The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges.

- Summary of the radius and interval of convergence for some power series already studied in this section.

	Series	Radius of Convergence	Interval of Convergence
Geometric Series	$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
Example 1	$\sum_{n=0}^{\infty} n!x^n$	$R = 0$	$\{0\}$
Example 2	$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$
Example 3	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$	$R = \infty$	$(-\infty, \infty)$

- **Example:** Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

SOLUTION The radius of convergence is $R = 1/3$. The interval of convergence is $(-1/3, 1/3]$.

- **Example:** Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n(n+2)^n}{3^{n+1}}$$

SOLUTION The radius of convergence is $R = 3$. The interval of convergence is $(-5, 1)$.

11.9 Representations of Functions as Power Series

- We can represent functions by power series. We have seen already the following.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n \quad |x| < 1$$

- **Example:** Express $1/(1+x^2)$ as the sum of a power series and find the interval of convergence.

  SOLUTION

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 \ldots$$

  The interval of convergence is $(-1, 1)$.

- **Example:** Find a power series representation for $1/(x+2)$.

  SOLUTION

$$\frac{1}{x+2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n$$

  The interval of convergence is $(-2, 2)$.

- **Example:** Find a power series representation for $x^3/(x+2)$.

  SOLUTION

$$\frac{x^3}{x+2} = \sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{2^{n-2}} x^n$$

  The interval of convergence is $(-2, 2)$.

- **Differentiation and Integration of Power Series**

- **Theorem:** If the power series $\sum c_n(x-a)^n$ has a radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

  is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

1. $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$
2. $\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \ldots$
   $= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

The radii of convergence of the power series in both 1 and 2 are R.

- We can write the equations in the above theorem as follows.

1. $\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}[c_n(x-a)^n]$
2. $\int \left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] dx = \sum_{n=0}^{\infty} \int c_n(x-a)^n\,dx$

- Although the radius of convergence R remains the same under integration and differentiation, the interval of convergence may change.

- **Example:** Express $1/(1-x)^2$ as a power series by differentiating the power series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n \quad |x| < 1.$$

What is the radius of convergence?

SOLUTION

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n, \quad R = 1$$

- **Example:** Find a power series representation for $\ln(1-x)$ and its radius of convergence.

SOLUTION

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}, \quad |x| < 1, \quad R = 1$$

Notice that we can find a power series for $\ln 2$ by using $\ln \frac{1}{2} = -\ln 2$.

- **Example:** Find a power series representation for $f(x) = \tan^{-1} x$ and its radius of convergence.

  SOLUTION

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad R = 1$$

- **Example:**

  1. Evaluate $\int [1/(1+x^7)]\, dx$ as a power series.

     SOLUTION

$$\int \frac{1}{(1+x^7)}\, dx = C + x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \ldots, \quad |x| < 1$$

  2. Use part 1 to approximate $\int_0^{0.5} [1/(1+x^7] dx$ correct to within 10^{-7}.

     SOLUTION Apply the Fundamental Theorem of Calculus the the above using $C = 0$.

$$\int_0^{0.5} \frac{1}{(1+x^7)}\, dx = \frac{1}{2} - \frac{1}{8 \cdot 2^8} + \frac{1}{15 \cdot 2^{15}} - \frac{1}{22 \cdot 2^{22}} + \ldots \frac{(-1)^n}{(7n+1)2^{7n+1}} + \ldots$$

     This is an alternating series. If we sum the first three terms only, the error will be less than the absolute value of the 4th term.

$$\frac{1}{29 \cdot 2^29} \approx 6.4 \times 10^{-11}$$

     Therefore, within an error of 10^{-7} we have

$$\int_0^{0.5} \frac{1}{(1+x^7)}\, dx = \frac{1}{2} - \frac{1}{8 \cdot 2^8} + \frac{1}{15 \cdot 2^{15}} - \frac{1}{22 \cdot 2^{22}} \approx 0.49951374 \quad \square$$

11.10 Taylor and MacLaurin Series

- The following formulas are derived by taking derivatives of a power series and evaluating at a.
- **Theorem** If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

- **The Taylor Series of the function f at a**

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

- In the case that $a = 0$, a Taylor series is called a Maclaurin series.
- **Example** Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.

SOLUTION $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\square$.

- Under what circumstances is a function equal to the sum of its Taylor series? That is, if f has derivatives of all orders, when is it true that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

- **Definition** The nth degree Taylor polynomial of f at a is

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

- In general, $f(x)$ is the sum of its Taylor series if

$$f(x) = \lim_{n\to\infty} T_n(x)$$

Let $R_n(x) = f(x) - T_n(x)$, which implies that $f(x) = T_n + R_n(x)$, then $R_n(x)$ is called the remainder of the Taylor series.

- **Theorem** If $f(x) = T_n(x) + R_n(x)$, where $T_n(x)$ is the nth degree Taylor polynomial of f at a and

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

- **Taylor's Inequality** If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| \leq d$$

The proof is in the book for the case where $n = 1$.

- We will use the following fact in later arguments:

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad \text{for every real number } x$$

This follows from the fact that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is convergent for all x.

- **Example** Prove that e^x is the sum of its Maclaurin series.

SOLUTION $f(x) = e^x \Rightarrow f^{(n+1)}(x) = e^x \forall n$. For any $d > 0$, if $|x| \leq d$, then $|f^{(n+1)}(x)| = e^x \leq e^d$. By Taylor's Inequality with $a = 0,\quad M = e^d$ gives

$$|R_n(x)| \leq \frac{e^d}{(n+1)!}|x|^{(n+1)} \quad \text{for } |x| \leq d$$

We now see that

$$\lim_{n\to\infty} \frac{e^d}{(n+1)!}|x|^{n+1} = e^d \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

Therefore, by the Squeeze Theorem $\lim_{n\to\infty} R_n(x) = 0 \ \Rightarrow \lim_{n\to\infty} R_n(x) = 0 \ \forall x$. We then can conclude therefore that e^x is equal to the sum of its Maclaurin series, i.e.,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \forall x \quad \blacksquare$$

- If we put $x = 1$ into the last equation we get

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$$

- **Example** Find the Taylor series for $f(x) = e^x$ at $x = 2$. SOLUTION $e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n, \quad \forall x \ \blacksquare$.

- **Example** Find the Maclaurin series for $\sin x$ and prove that it represents $\sin x$ for all x.

$$\begin{aligned}
f(x) &= \sin x \quad f(0) = 0 \\
f'(x) &= \cos x \quad f'(0) = 0 \\
f''(x) &= -\sin x \quad f''(0) = 0 \\
f'''(x) &= -\cos x \quad f'''(0) = 0 \\
f^{(4)}(x) &= \sin x \quad f^{(4)}(0) = 0
\end{aligned}$$

The derivatives repeat in cycles of four. Therefore, we can conclude that the Maclaurin series is

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

To show that the remainder goes to 0, note that $|\sin x| \leq 1$ and $|\cos x| \leq 1$, so we can take $M = 1$ in Taylor's Inequality:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x^{n+1}| = \frac{|x|^{n+1}}{(n+1)!}$$

The right side of the inequality goes to 0, and therefore, by the Squeeze Theorem, $R_n(x)$ goes to 0. Therefore $\sin x$ equals its Taylor series. ■

- **Example** Find the Maclaurin series for $\cos x$.

  The easiest way to do this is to differentiate the formula for $\sin x$.

$$\begin{aligned}\cos x &= \frac{d}{dx}(\sin x) = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right) \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \quad \text{for all} x \quad \blacksquare\end{aligned}$$

- **Example** Find the Maclaurin series for the function $f(x) = x\cos x$.

  SOLUTION Multiply the series for $\cos x$ by x:

$$x\cos x = x\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n)!}$$

- **Example** Represent $f(x) = \sin x$ as the sum of the Taylor series centered at $\pi/3$.

  SOLUTION

$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n\sqrt{3}}{2(2n)!}\left(x - \frac{\pi}{3}\right)^{2n} + \sum_{n=0}^{\infty}\frac{(-1)^n}{2(2n+1)!}\left(x - \frac{\pi}{3}\right)^{2n+1} \quad \blacksquare$$

- Summary of important Maclaurin series of page 767.

- **Example**

1. Evaluate $\int e^{-x^2}\,dx$ as an infinite series.

   SOLUTION First, find the Maclaurin series for $f(x) = e^{-x^2}$. We can substitute $-x^2$ into the Maclaurin series for e^x. We get

   $$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - \frac{x^2}{2!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots$$

   Integrating gives

   $$\int e^{-x^2}\,dx = \int \left(1 - \frac{x^2}{2!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots + (-1)^n \frac{x^{2n}}{n!} + \ldots\right) dx$$

   $$= C + x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)n!} + \ldots$$

   The series converges for all x because the original series for e^{-x^2} converges for all x.

2. Evaluate $\int e^{-x^2}\,dx$ correct to within an error of 0.001.

   SOLUTION

   $$\begin{aligned} \int_0^1 e^{-x^2}\,dx &= \left[x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} + \cdots\right]_0^1 \\ &= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \cdots \\ &\approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} \approx 0.7475 \end{aligned}$$

   The Alternating Series Estimation Theorem shows that the error in this approximation is less that

   $$\frac{1}{11 \cdot 5!} = \frac{1}{1320} < 0.001 \quad \blacksquare$$

- Evaluate $\lim_{n\to\infty} \frac{e^x - 1 - x}{x^2}$.

  SOLUTION Using the Maclaurin series for e^x, (note that power series are continuous function), we have

$$\begin{aligned}
\lim_{x\to 0}\frac{e^x-1-x}{x^2} &= \lim_{x\to 0}\frac{\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)-1-x}{x^2}\\
&= \lim_{x\to 0}\frac{\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots}{x^2}\\
&= \lim_{x\to 0}\left(\frac{1}{2}+\frac{x}{3!}+\frac{x^2}{4!}+\frac{x^5}{5!}+\ldots\right)\\
&= \frac{1}{2} \quad \blacksquare
\end{aligned}$$

- **Multiplication and Division of Power Series**

  Power series can be multiplied and divided like polynomials.

- **Example** Find the first three nonzero terms in the Maclaurin series for

  1. $e^x \sin x$

     SOLUTION Using the Maclaurin series for e^x and $\sin x$ gives

     $$e^x \sin x = \left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)\left(x-\frac{x^3}{3!}+\ldots\right)$$

     Multiplying gives

     $$e^x \sin x = x + x^2 + \frac{1}{3}x^3 + \ldots$$

  2. $\tan x$

     SOLUTION

     $$\tan x = \frac{\sin x}{\cos x} = \frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots}{1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots}$$

     We can do long division

     $$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots \quad \blacksquare$$

- Homework §11.10 #1-17, 23-31, 39-59

11.11 The Binomial Series

- **The Binomial Theorem** If a and b are any real numbers and k is a positive integer, then

$$(a+b)^k = \sum_{n=0}^{k} \binom{k}{n} a^{k-n} b^n$$

Where

$$\begin{aligned} \binom{k}{n} &= \frac{k!}{n!(k-n)!} \\ &= \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}, \quad n = 1, 2, \ldots, k \end{aligned}$$

If we let $a = 1$ and $b = x$, then we get

$$(1+x)^k = \sum_{n=0}^{k} \binom{k}{n} x^n$$

Newton extended the Binomial Theorem to the case in which k is not a positive integer. In this case, the expansion is an infinite sum. We can find the infinite sum be finding the Maclaurin series of $f(x) = (1+x)^k$.

In summary,

$$f^{(n)}(x) = k(k-1)(k-2)\cdots(k-n+1)(1+x)^{k-n}$$

$$\text{and} \qquad f^{(n)}(0) = k(k-1)\cdots(k-n+1)$$

Therefore the Maclaurin series of $f(x) = (1+x)^k$ is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} x^n$$

The series is called the **binomial series**. We can use the Ratio Test to find that

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|k-n|}{n+1}|x| = \frac{\left|1-\frac{k}{n}\right|}{1+\frac{1}{n}} |x| \to |x| \quad \text{as } n \to \infty$$

Therefore, by the Ratio Test, the binomial series converges if $|x| < 1$ and diverges if $|x| > 1$. It can be proven that the function $(1+x)^k$ is equal to the sum of its Maclaurin series. This is proven by showing that $R_n(x) \to 0$ as $n \to \infty$, but this is not easy.

- **The Binomial Series** If k is any real number and $|x| < 1$, then

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \ldots = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

where

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} \quad n \geq 1 \text{ and } \binom{k}{0} = 1$$

- **Example** Expand $\frac{1}{(1+x)^2}$ as a power series.

SOLUTION We use the binomial series with $k = 2$.

$$\begin{aligned}\binom{-2}{n} &= \frac{(-2)(-3)(-4)\cdots(-2-n+1)}{n!} \\ &= \frac{(-1)^n 2 \cdot 3 \cdot 4 \cdots n(n+1)}{n!} \\ &= (-1)^n(n+1)\end{aligned}$$

Therefore, for $|x| < 1$,

$$\begin{aligned}\frac{1}{(1+x)^2} &= (1+x)^{-2} = \sum_{n=0}^{\infty} \binom{-2}{n} x^n \\ &= \sum_{n=0}^{\infty} (-1)^n (n+1) x^n = 1 - 2x + 3x^2 - 4x^3 + \cdots \quad \blacksquare\end{aligned}$$

- **Example** Find the Maclaurin series for the function $f(x) = \frac{1}{\sqrt{4-x}}$ and it radius of convergence.

SOLUTION We rewrite the function

$$\frac{1}{\sqrt{4-x}} = \frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2}$$

Then we can use the Binomial Theorem with $k = -\frac{1}{2}$ with x replaced by $-x/4$ and get

$$\begin{aligned}
\frac{1}{\sqrt{4-x}} &= \frac{1}{2}\left(1-\frac{x}{4}\right)^{-1/2} = \frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(-\frac{x}{4}\right)^n \\
&= \frac{1}{2}\left[1+\left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right)+\frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\left(-\frac{x}{4}\right)^2+\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}\left(-\frac{x}{4}\right)^3\right. \\
&+ \left.\cdots+\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{1}{2}-n+1)}{n!}\left(-\frac{x}{4}\right)^n+\cdots\right] \\
&= \frac{1}{2}\left[1+\frac{1}{8}x+\frac{1\cdot 3}{2!8^2}x^2+\frac{1\cdot 3\cdot 5}{3!8^3}x^3+\cdots+\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!8^n}x^n+\cdots\right]
\end{aligned}$$

The series converges for $|-x/4| < 1$, i.e., $|x| < 4$, so the radius of convergence is $R = 4$. ■

- Homework §11.11 #1-7, 11-17

A Course Content and Scope/Topic Outline

- Techniques of Integration
  - Computing definite and indefinite integrals using integration by parts
  - Computing definite and indefinite integrals of products and powers of trig functions
  - Computing definite and indefinite integrals using the method of trigonometric substitutions
  - Computing definite and indefinite integrals of rational functions using the method of partial fractions

  - Computing definite and indefinite integrals using a rationalizing substitution
  - Approximating definite integrals using the trapezoid rule and Simpson's rule and the accompanying analysis of bounds for the errors in these approximations
  - Computing improper integrals with infinite limits of integration
  - Computing improper integrals with discontinuous integrands

- More Applications of Integration
  - Computing the length of a curve
  - Computing the area of a surface of revolution
  - Computing moments and centers of mass in one and two dimensions

- Differential Equations
  - Solving separable differential equations with or without initial conditions
  - Solving first order homogeneous differential equations
  - Solving first order linear differential equations using the integrating factor method

- Parametric Equations and Polar Coordinates
  - Standard parameterizations of simple conics and cycloids
  - Graphing simple parametric curves
  - Computing first and second derivative of y with respect to x for curves given parametrically
  - Computing the length of a curve given parametrically
  - Computing the surface area of revolution for a parametric curve revolved about a horizontal or vertical line
  - Graphing in polar coordinates
  - Finding tangents to polar curves
  - Computing areas and curve length in polar coordinates

  - Conic sections in Cartesian form

- Infinite Sequences and Series
  - Computing limits of sequences using properties of same
  - Discussion of the definitions of limits of sequences
  - Discussion of monotone sequences and bounds for sequences
  - Definition of a series as the limit of its sequence of partial sums
  - Computing with geometric series
  - Discussion of properties of convergent series
  - Discussion and use of the Nth Term Test and p-series
  - Discussion and use of the Comparison Test, the Integral Test, and the Limit Comparison Test for series with non-negative terms
  - Discussion of absolute convergence and conditional convergence for series
  - Discussion and use of the Alternating Series Test and truncation errors
  - Discussion and use of the Nth Root Test and the Ratio Test for series
  - Discussion of power series and the interval of convergence Constructing Taylor Series and Maclaurin Series
  - Discussion of Taylor's Formula and using Taylor and Maclaurin Series for approximations to functions, complete with the accompanying error analysis
  - Discussion and use of binomial series

B INSTRUCTIONAL OBJECTIVES:

The student will be able to:

1. Calculate definite and indefinite integrals using simple substitutions, integrating by parts, trigonometric substitutions, and partial fractions.

2. Use integration to solve a variety of applications including areas, volumes of revolution, surface areas of revolution, lengths of curves, and centers of mass.

3. Calculate derivatives and compute integrals to solve applications given by functions in polar or parametric form.

4. Compute limits of sequences.

5. Apply tests for convergence or divergence of series, distinguishing among divergence, absolute convergence, and conditional convergence.

6. Construct and analyze Taylor and Maclaurin series.

7. Solve simple first order differential equations.

C Topics, Sections

- Techniques of Integration
  - §7.1 Integration by Parts
  - §7.2 Trigonometric Integrals
  - §7.3 Trigonometric Substitutions
  - §7.4 Integration of Rational Functions by Partial Fractions and Rationalizing Substitution
  - §7.7 Approximate Integration
  - §7.8 Improper Integrals
- More Applications of Integration
  - §8.1 Arc Length
  - §8.2 Area of a Surface of Revolution
  - §8.3 Applications to Physics and Engineering: moments and centers of mass
- Differential Equations
  - §9.3 Separable Equations

  - §9.6 Linear Equations

- Parametric Equations and Polar Coordinates
  - §10.1 Curves Defined by Parametric Equations
  - §10.2 Calculus with Parametric Curves
  - §10.3 Polar Coordinates
  - §10.4 Areas and Lengths in Polar Coordinates
  - §10.5 Conic Sections

- Infinite Sequences and Series
  - §11.1 Sequences
  - §11.2 Series
  - §11.3 The Integral Test and Estimate of Sums
  - §11.4 The Comparison Tests
  - §11.5 Alternating Series
  - §11.6 Absolute Convergence and the Ratio and Root Tests
  - §11.7 Strategy for Testing Series
  - §11.8 Power Series
  - §11.9 Representations of Functions as Power Series
  - §11.10 Taylor and MacLaurin Series
  - §11.11 The Binomial Series