Math 185 Tues 2/09

Homework Review

§7.3 #15 $\int_0^a x^2\sqrt{a^2-x^2}\,dx$

$$x = a\sin\theta$$
$$dx = a\cos\theta\,d\theta$$
$$\sqrt{a^2-x^2} = \sqrt{a^2-(a\sin\theta)^2}$$
$$= \sqrt{a^2(1-\sin^2\theta)}$$
$$= \sqrt{a^2\cos^2\theta} = a\cos\theta$$

Bounds of Integration.

- when $x=0$, $0 = a\sin\theta$, $\sin\theta = 0$, $\theta = 0$
- when $x=a$, $a = a\sin\theta$, $\sin\theta = 1$, $\theta = \pi/2$

$$= \int_0^{\pi/2} (a\sin\theta)^2\cdot a\cos\theta\cdot a\cos\theta\,d\theta$$
$$= a^4\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta$$

Note
$\sin 2\theta = 2\sin\theta\cos\theta$
$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$

$$= a^4\int_0^{\pi/2}(\sin\theta\cos\theta)^2\,d\theta$$
$$= a^4\int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^2 d\theta$$
$$= \frac{a^4}{4}\int_0^{\pi/2}\sin^2(2\theta)\,d\theta$$

Use $\sin^2 u = \frac{1}{2}(1-\cos 2u)$

$$= \frac{a^4}{4}\int_0^{\pi/2}\frac{1}{2}(1-\cos 4\theta)\,d\theta$$

$$= \frac{a^4}{8}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2}$$

$$= \frac{a^4}{8}\left[\frac{\pi}{2} - \frac{1}{4}\underbrace{\sin\left(4\cdot\frac{\pi}{2}\right)}_{\sin 2\pi = 0}\right] - \frac{a^4}{8}\left[0 - \frac{1}{4}\underbrace{\sin(4\cdot 0)}_{\sin(0)=0}\right]$$

$$= \frac{a^4}{8}\cdot\frac{\pi}{2} = \frac{a^4\pi}{16}$$

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§7.1, 7.2

Know these identities

$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\sin^2\theta + \cos^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

$\int \tan x = \ln|\sec x| + C$

$\int \sec x = \ln|\sec x + \tan x| + C$

§7.1 #13 $\int t\sec^2(2t)\,dt$

$w = 2t,\quad t = \frac{w}{2}$

$dw = 2\,dt$

$\frac{1}{2}dw = dt$

$\frac{1}{2}\int \frac{w}{2}\sec^2(w)\,dw$

$\frac{1}{4}\int w\sec^2 w\,dw$

Int. by Parts

$u = w \qquad dv = \sec^2 w\,dw$

$du = dw \qquad v = \tan w$

$\int u\,dv = uv - \int v\,du$

$= \frac{1}{4}(w\tan w - \int \tan w\,dw)$

$= \frac{1}{4}w\tan w - \frac{1}{4}\ln|\sec w| + C$

$= \frac{1}{4}(2t)\tan 2t - \frac{1}{4}\ln|\sec 2t| + C$

$= \boxed{\frac{t}{2}\tan 2t - \frac{1}{4}\ln|\sec 2t| + C}$

§7.1 Practice.
Use Integration by Parts to evaluate.

$$\int e^{3x} \sin x \, dx$$

$u = e^{3x}$ $\quad$ $dv = \sin x \, dx$

$du = 3e^{3x}$ $\quad$ $v = -\cos x$

$$\int u \, dv = uv - \int v \, du = -e^{3x}\cos x - \int -3e^{3x}\cos x \, dx$$

$$= -e^{3x}\cos x + 3\int e^{3x}\cos x \, dx$$

$u = e^{3x}$ $\quad$ $dv = \cos x \, dx$

$du = 3e^{3x}$ $\quad$ $v = \sin x$

$$= e^{3x}\sin x - \int 3e^{3x}\sin x \, dx$$

$$\int e^{3x} \overset{I}{\sin} x \, dx = -e^{3x}\cos x + 3\left(e^{3x}\sin x - 3\int e^{3x} \overset{I}{\sin} x \, dx\right)$$

$$I = -e^{3x}\cos x + 3e^{3x}\sin x \overset{-9I}{-9\int e^{3x}\sin x \, dx}$$

$$\overset{10I}{10\int e^{3x}\sin x \, dx} = -e^{3x}\cos x + 3e^{3x}\sin x$$

$$\int e^{3x}\sin x \, dx = \frac{e^{3x}}{10}(3\sin x - \cos x) + C$$

§7.1 Practice Problem

② $\int x^2 \sin 7x \, dx$

$u = x^2 \qquad dv = \sin 7x \, dx$

$du = 2x \, dx \qquad v = -\frac{1}{7} \cos 7x$

$= -\frac{1}{7} x^2 \cos 7x - \int -\frac{1}{7} \cdot 2x \cos 7x \, dx$

$= -\frac{1}{7} x^2 \cos 7x + \frac{2}{7} \int x \cos 7x \, dx$

$u = x \qquad dv = \cos 7x \, dx$

$du = dx \qquad v = \frac{1}{7} \sin 7x$

$= \frac{x}{7} \sin 7x - \int \frac{1}{7} \sin 7x \, dx$

$= \frac{x}{7} \sin 7x + \frac{1}{49} \cos 7x$

$= -\frac{1}{7} x^2 \cos 7x + \frac{2}{7}\left(\frac{x}{7} \sin 7x + \frac{1}{49} \cos 7x \right) + C$

$= -\frac{1}{7} x^2 \cos(7x) + \frac{2x}{49} \sin(7x) + \frac{2}{343} \cos(7x) + C$

§7.4 Integration of Rational Functions using Partial Fractions.

Explanation:

$$\frac{2}{x+1} - \frac{1}{x+2} = \frac{2\,(x+2)}{(x+1)(x+2)} - \frac{(x+1)\cdot 1}{(x+1)\;x+2}$$

$$= \frac{2x+4-x-1}{(x+1)(x+2)} = \frac{x+3}{(x+1)(x+2)}$$

$$= \frac{x+3}{x^2+3x+2}$$

Suppose we are asked to evaluate $\displaystyle\int \frac{x+3}{x^2+3x+2}\,dx$

Use: $\displaystyle = \int \frac{2}{x+1}\,dx - \int \frac{1}{x+2}\,dx$

$$= \boxed{2\ln|x+1| - \ln|x+2| + C}$$

$$\boxed{\begin{aligned} \int \frac{2}{x+1}\,dx &= \int \frac{2}{u}\,du = 2\ln|u| + C \\ &= 2\ln|x+1| + C \\ u &= x+1 \\ du &= dx \end{aligned}}$$

Example: $\int \frac{x+1}{x^2+5x+6}\,dx$

$$\frac{x+1}{x^2+5x+6} = \frac{x+1}{(x+2)(x+3)}$$

$$\frac{x+1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3} \qquad (*)$$

Solve for A and B

Clear the fractions. Multiply through by $(x+2)(x+3)$

$$\frac{(x+1)}{(x+2)(x+3}\,(x+2)(x+3) = \frac{A}{(x+2)}(x+2)(x+3) + \frac{B}{(x+3)}(x+2)(x+3)$$

$$x+1 = A(x+3) + B(x+2)$$

- Let $x=-3$

$$(-3)+1 = A(-3+3) + B(-3+2)$$
$$-2 = A(0) + B(-1)$$
$$-2 = -B$$
$$\boxed{B=2}$$

- Let $x=-2$

$$(-2)+1 = A(-2+3) + B(-2+2)$$
$$-1 = A(1) + B(0)$$
$$-1 = A$$
$$\boxed{A=-1}$$

We have from $(*)$

$$= \int \frac{-1}{x+2} + \frac{2}{x+3}\,dx = -\ln|x+2| + 2\ln|x+3| + C.$$