Math 185 Weds Feb. 10, 2010

§7.3 #25

$$\int \frac{x}{\sqrt{x^2+x+1}}\,dx$$

Complete the square

$$x^2+x+1 = \left(x^2+x+\tfrac{1}{4}\right)-\tfrac{1}{4}+1$$

$$\left(\tfrac{1}{2}\right)^2$$

$$=\left(x+\tfrac{1}{2}\right)^2+\tfrac{3}{4}$$

$$\int \frac{x}{\sqrt{(x+1/2)^2+3/4}}\,dx$$

$u = x+1/2$, $x = u-1/2$

$du = dx$

$$\int \frac{u-1/2}{\sqrt{u^2+3/4}}\,du$$

$a^2 = \frac{3}{4}$

$a = \frac{\sqrt{3}}{2}$

$u = \frac{\sqrt{3}}{2}\tan\theta$

$du = \frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$

$$\sqrt{u^2+3/4} = \sqrt{\left(\tfrac{\sqrt{3}}{2}\tan\theta\right)^2+\tfrac{3}{4}}$$

$$= \sqrt{\tfrac{3}{4}\left(\tan^2\theta+1\right)}$$

$$= \tfrac{\sqrt{3}}{2}\sqrt{\sec^2\theta}$$

$$= \tfrac{\sqrt{3}}{2}\sec\theta$$

$$= \int \frac{\left(\frac{\sqrt{3}}{2}\tan\theta-\frac{1}{2}\right)\cdot\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta}{\frac{\sqrt{3}}{2}\sec\theta}$$

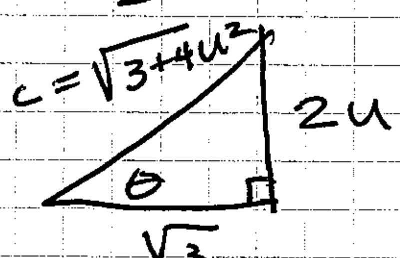
$$= \int\left(\tfrac{\sqrt{3}}{2}\tan\theta-\tfrac{1}{2}\right)\sec\theta\,d\theta$$

$$= \int\left(\tfrac{\sqrt{3}}{2}\tan\theta\sec\theta-\tfrac{1}{2}\sec\theta\right)d\theta$$

$$= \tfrac{\sqrt{3}}{2}\sec\theta-\tfrac{1}{2}\ln\left|\sec\theta+\tan\theta\right|+C$$

Substitute back for ~~x~~ u.

$u = \frac{\sqrt{3}}{2}\tan\theta$, $\tan\theta = \frac{2u}{\sqrt{3}} = \frac{\text{opp}}{\text{adj}}$



$\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{3+4u^2}}{\sqrt{3}}$

$c^2 = (\sqrt{3})^2+(2u)^2$

$c^2 = 3+4u^2$, $c = \sqrt{3+4u^2}$

We get:

$$= \frac{\sqrt{3}}{2}\frac{\sqrt{3+4u^2}}{\sqrt{3}} - \frac{1}{2}\ln\left|\frac{\sqrt{3+4u^2}}{\sqrt{3}} + \frac{2u}{\sqrt{3}}\right| + C$$

$$= \frac{\cancel{\sqrt{3}}}{\cancel{2}}\frac{\sqrt{\cancel{4}(x^2+x+1)}}{\cancel{\sqrt{3}}} - \frac{1}{2}\ln\left|\frac{\sqrt{4(x^2+x+1)}}{\sqrt{3}} + \frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right| + C$$

Put in $u = x + \frac{1}{2}$

$$3+4u^2 = 3+4\left(x+\frac{1}{2}\right)^2$$
$$= 3 + 4\left(x^2+x+\frac{1}{4}\right)$$
$$= 3+4x^2+4x+1$$
$$= 4x^2+4x+4$$
$$= 4(x^2+x+1)$$

$$= \sqrt{x^2+x+1} - \frac{1}{2}\ln\left|\frac{2}{\sqrt{3}}\cdot\left(\sqrt{x^2+x+1} + x + \frac{1}{2}\right)\right| + C$$

put with C

$$= \sqrt{x^2+x+1} - \frac{1}{2}\left(\ln\frac{2}{\sqrt{3}} + \ln\left|\sqrt{x^2+x+1} + x+\frac{1}{2}\right|\right) + C$$

$\ln AB = \ln A + \ln B$

$$= \sqrt{x^2+x+1} - \frac{1}{2}\ln\left|\sqrt{x^2+x+1} + x + \frac{1}{2}\right| + C$$

§7.4 continued

Example Evaluate $\int \frac{x+2}{x^2-x-2}\,dx$

Use partial fractions.

$$\frac{x+2}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

SOLVE FOR A AND B.

Clear the fraction.

Multiply both sides by $(x-2)(x+1)$.

$$x+2 = A(x+1) + B(x-2)$$

- $x=2$

$$(2)+2 = A(2+1) + B(2-2)$$
$$4 = 3A$$
$$\boxed{A = 4/3}$$

- $x=-1$

$$(-1)+2 = A(-1+1) + B(-1-2)$$
$$1 = -3B$$
$$\boxed{B = -1/3}$$

Here's another way to solve for A & B

$$x+2 = A(x+1) + B(x-2) \quad \text{(expand)}$$
$$x+2 = Ax + A + Bx - 2B$$
$$x+2 = (Ax+Bx) + (A-2B) \quad \text{(group like terms)}$$
$$1x+2 = (A+B)x + (A-2B)$$

The coefficients of x on the Left Hand Side must equal the coefficients of x on the Right Hand Side.

$$1 = A+B \longrightarrow 2A+2B=2$$
$$2 = A-2B \qquad\quad A-2B=2$$

$$3A=4$$
$$\boxed{A = 4/3}$$

$$A+B=1$$
$$4/3+B=1$$
$$\boxed{B=-1/3}$$

We get

$$\int \left(\frac{A}{x-2} + \frac{B}{x+1} \right) dx$$

$$= \int \frac{4}{3} \cdot \left(\frac{1}{x-2} \right) + \frac{-1}{3} \left(\frac{1}{x+1} \right) dx$$

$$= \frac{4}{3} \ln|x-2| - \frac{1}{3} \ln|x+1| + C$$

<u>Example</u>: $\int \frac{x^2}{x-1} \, dx$

If the degree of the numerator is greater than or equal to the degree of the denominator, then first do long division.

$$\begin{array}{r l}
 & x + 1 + \frac{-1}{x-1} \\
x-1 \,\big) & x^2 + 0x + 0 \\
 & -(x^2 - x) \downarrow \\
\hline
 & 0 + x + 0 \\
 & -x \cancel{+} 1 \\
\hline
 & 1
\end{array}$$

$\div, \times, -, \downarrow$

$$= \int \left(x + 1 + \frac{1}{x-1} \right) dx = \frac{x^2}{2} - x + \ln|x-1| + C$$

Example: $\int \frac{x+3}{x^2+2x+1}\,dx$

$$\frac{x+3}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

Solve for A & B

- Multiply through by $(x+1)^2$

$$x+3 = A(x+1) + B$$

- $x = -1$

$$-1+3 = A(-1+1) + B$$
$$2 = B$$
$$\boxed{B=2}$$

- $x=0$

$$0+3 = A(0+1) + B$$
$$3 = A + B$$
$$3 = A + 2 \leftarrow B=2$$
$$\boxed{A=1}$$

$$= \int \left(\frac{1}{x+1} + \frac{2}{(x+1)^2} \right) dx$$

$$= \int \left(\frac{1}{x+1} + 2(x+1)^{-2} \right) dx$$

$$= \ln|x+1| + 2\frac{(x+1)^{-1}}{-1} + C$$

$$= \ln|x+1| - \frac{2}{x+1} + C \quad \square$$

Example $\int \frac{2x^2+3x+2}{x^3+x}\,dx$

$$\frac{2x^2+3x+2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

irreducible:

$$\boxed{\begin{aligned} x^2+1&=0 \\ x^2&=-1 \\ x&=\pm\sqrt{-1}=\pm i \end{aligned}}$$

Solve for A, B, & C.

Multiply through by $x(x^2+1)$.

$$2x^2+3x+2 = A(x^2+1) + (Bx+C)x$$

- $x=0$

$$2 = A(0^2+1) + (B\cdot 0 + C)\cdot 0$$

$$\boxed{2 = A}$$

put $A=2$ back into this & simplify

$$2x^2+3x+2 = 2(x^2+1) + (Bx+C)x$$

$$\cancel{2x^2}+3x+\cancel{2} = \cancel{2x^2}+\cancel{2} + (Bx+C)x$$

$$3x = (Bx+C)x$$

- $x=1$

$$3(1) = (B\cdot 1 + C)1$$

$$3 = B+C$$

- $x=-1$

$$3(-1) = (B(-1)+C)(-1)$$

$$-3 = (-B+C)(-1)$$

$$-3 = B-C$$

$$\begin{array}{r} B+C=3 \\ B-C=-3 \\ \hline 2B=0 \end{array}$$

$$\boxed{B=0}$$

$\Rightarrow$

$$B+C=3$$

$$0+C=3$$

$$\boxed{C=3}$$

$$\int \left(\frac{A}{x} + \frac{Bx+C}{x^2+1} \right) dx$$

$$= \int \left(\frac{2}{x} + \frac{0 \cdot x+3}{x^2+1} \right) dx$$

$$= \int \left(\frac{2}{x} + \frac{3}{x^2+1} \right) dx$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{x^2+1}$$

$$= \boxed{2 \ln|x| + 3 \tan^{-1} x + C}$$

FORMULA

$$\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$