§7.4 Continued

Math 185
Thurs. 2/11/2010

Example: An integral of a rational function where the denominator IS an irreducible quadratic.

$$\int \frac{2x+5}{x^2+9}\,dx$$

$$= \int \left(\frac{2x}{x^2+9} + \frac{5}{x^2+9} \right) dx$$

(I) (II)

(I) $\int \frac{2x}{x^2+9}\,dx$

Use subst. method

$u = x^2+9$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$\frac{1}{2}\int \frac{2}{u}\,du = \ln|u| + C_1$$
$$= \ln|x^2+9| + C_1$$

(II) $\int \frac{5}{x^2+9}\,dx$

Use: $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

$$= 5\int \frac{1}{x^2+3^2}\,dx \leftarrow a=3$$

$$= \cancel{5 \cdot \frac{1}{3}\int 1} \quad \cancel{= 5\,1}$$
$$= 5\left(\frac{1}{3}\right)\tan^{-1}\left(\frac{x}{3}\right) + C_2$$

Answer: $\ln|x^2+9| + \frac{5}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$

Example: Again, the dominator is irreducible.

$$\int \frac{7x+6}{x^2+4x+13}\,dx$$

Complete the square. of the denom.

$x^2+4x+13$

$$= (x^2+4x+4)-4+13$$

$$\left(\tfrac{4}{2}\right)^2$$

$$= (x+2)^2+9$$

$$= \int \frac{7x+6}{(x+2)^2+9}\,dx$$

$u = x+2$, $\Rightarrow u-2=x$

$du = dx$

$$= \int \frac{7(u-2)+6}{u^2+9}\,du$$

$$= \int \frac{7u-14+6}{u^2+9}\,du$$

$$= \int \frac{7u-8}{u^2+9}\,du$$

$$= \int \frac{7\overset{\mathrm{I}}{u}}{u^2+9}\,du - 8\overset{\mathrm{II}}{\int}\frac{1}{u^2+9}\,du$$

Try to factor $x^2+4x+13$.

$a=1$, $b=4$, $c=13$

Quad formula

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-4\pm\sqrt{4^2-4\cdot1\cdot13}}{2\cdot1}$$

$$= \frac{-4\pm\sqrt{16-52}}{2}$$

$$= \frac{-4\pm\sqrt{-36}}{2}$$

$$= \frac{-4\pm6i}{2}$$

imaginary. Does not factor over the reals.

Shorter:

Discriminant: if $b^2-4ac<0$, then irreducible

(I)

$$= 7\int \frac{u}{u^2+9}\,du$$

$w = u^2+9$

$dw = 2u\,du$

$\frac{1}{2}dw = u\,du$

$$= \frac{7}{2}\int \frac{1}{w}\,dw = \frac{7}{2}\ln|w| + C_1$$

$$= \frac{7}{2}\ln|\cancel{x}\, u^2+9| + C_1$$

$$= \frac{7}{2}\ln|x^2+4x+13| + C_1$$

(II) $$-8\int \frac{1}{u^2+9}\,du = -8\cdot\frac{1}{3}\tan^{-1}\frac{u}{3} + C_2$$

$$= \frac{-8}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + C_2$$

<u>Answer:</u> $\frac{7}{2}\ln|x^2+4x+13| - \frac{8}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + C$

Here are the ~~4~~ 3 cases for partial fraction decomposition of a rational function, $\frac{P(x)}{Q(x)}$.

$\deg P < \deg Q$.

CASE I: $Q(x)$ is a product of distinct linear factors.

$$Q(x) = (a_1 x + b_1)(a_2 x + b_2) \cdots (a_k x + b_k)$$

Then

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \cdots + \frac{A_k}{a_k x + b_k}$$

Example $\displaystyle\int \frac{x^2 + 5x + 5}{(x-1)(x-2)(x-3)}\, dx$

$$\frac{x^2 + 5x + 5}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}.$$

CASE II $Q(x)$ is a product of repeated linear factors.

$$Q(x) = (ax+b)^r$$

$$\frac{P(x)}{Q(x)} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots \frac{A_r}{(ax+b)^r}$$

Example: $\int \frac{x^2+5x+7}{(x-2)^3}\,dx$

$$\frac{x^2+5x+7}{(x-2)^3} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3}$$

CASE III $Q(x)$ contains a irreducible quadratic factor, say

$Q(x) = ax^2+bx+c$,

$b^2-4ac<0$

then the partial term looks like $\frac{Ax+B}{ax^2+bx+c}$

Example $\int \frac{x^2+5x+2}{(x+1)^2(x^2+4)}\,dx$

$$\frac{x^2+5x+2}{(x+1)^2(x^2+4)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+4}$$

CASE IV: $Q(x)$ contains repeated irreducible quadratic factors.

$$Q(x) = (ax^2+bx+c)^r$$

$$b^2-4ac<0$$

$$\frac{P(x)}{Q(x)} = \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots \frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

Example: $\dfrac{x^5+7x+2}{(x^2+9)^3}$

$$\frac{x^5+7x+2}{(x^2+9)^3} = \frac{Ax+B}{x^2+9} + \frac{Cx+D}{(x^2+9)^2} + \frac{Ex+F}{(x^2+9)^3}$$

Example $\dfrac{x^7+3x^5+9}{x^2(x^2+4)^3}$

x^2 repeated linear; $(x^2+4)^3$ repeated quadratic

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$$= \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+4} + \frac{Ex+F}{(x^2+4)^2} + \frac{Gx+H}{(x^2+4)^3}$$

#6

Rationalizing Substitutions

Example: Evaluate $\int \frac{\sqrt{x+4}}{x}\,dx$

~~we are not going to~~

We let $u = \sqrt{x+4}$

rule of thumb: let u equal the radical expression.

$u^2 = x+4$

$2u\,du = dx$

~~we at~~

We have

$u^2 = x+4$

$x = u^2 - 4$

$$\int \frac{u}{u^2-4} \cdot 2u\,du = \int \frac{2u^2}{u^2-4}\,du$$

deg num = deg denom

$$\begin{array}{r|l} & 2 + \frac{8}{u^2-4} \\ \hline u^2-4 & 2u^2+0u+0 \\ & -2u^2 \quad +8 \\ \hline & 8 \end{array}$$

$$= \int \left(\overset{\text{I}}{2} + \overset{\text{II}}{\frac{8}{u^2-4}} \right) du$$

I $\int 2\,du = 2u + C_1$

$= 2\sqrt{x+4} + C_1$

II $\displaystyle\int \frac{8}{u^2-4}\,du = \int \frac{8}{(u-2)(u+2)}\,du$

$$\frac{8}{(u-2)(u+2)} = \frac{A}{u-2} + \frac{B}{u+2}$$

$$8 = A(u+2) + B(u-2)$$

- $u = -2$

  $8 = A(-2+2) + B(-2-2)$

  $-4B = 8$, $\boxed{B = -2}$

- $u = 2$

  $8 = A(2+2) + B(2-2)$

  $4A = 8$, $\boxed{A = 2}$

$$\int\left(\frac{2}{u-2} + \frac{-2}{u+2}\right)du = 2\ln|u-2| - 2\ln|u+2| + C_2$$

$$= 2\ln\left|\sqrt{x+4} - 2\right| - 2\ln\left|\sqrt{x+4}+2\right| + C_2$$

Answer: $2\sqrt{x+4} + 2\ln\left|\sqrt{x+4} - 2\right| - 2\ln\left|\sqrt{x+4}+2\right| + C$

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9