Math 185 Weds. 2/17/2010

§7.7 Approximate Integration

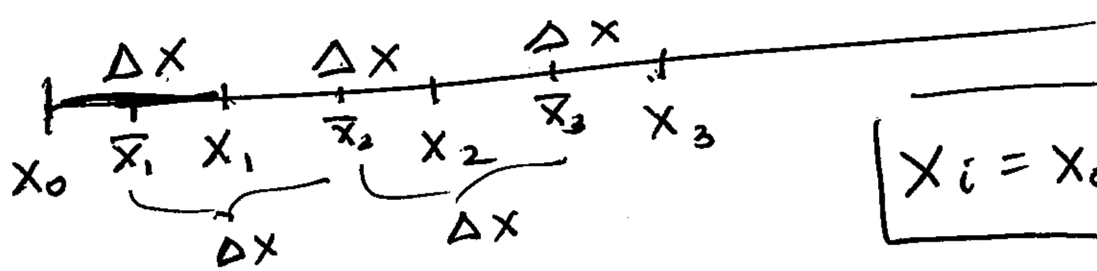
Midpoint Rule

$$\int_a^b f(x)\,dx \approx M_n = \Delta x\left[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)\right]$$

$$\Delta x = \frac{b-a}{n}$$

$$\bar{x}_i = \frac{x_{i-1} + x_i}{2} = \text{midpoint of } [x_{i-1}, x_i]$$

Note



$$\boxed{x_i = x_0 + i\,\Delta x}$$

$$x_1 = \frac{x_0 + x_1}{2}$$

$$\boxed{x_i = x_{i-1} + \Delta x}$$

Example Use the Midpoint Rule with $n=5$ to approximate

$$\int_1^2 \frac{1}{x}\,dx$$

Note $\displaystyle\int_1^2 \frac{1}{x}\,dx = \ln|x|\Big]_1^2$

$$= \ln 2 - \underbrace{\ln 1}_{=0}$$

$$= \ln 2.$$

SOLUTION

$n = 5$

$a = 1$

$b = 2$

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{5} = \frac{1}{5} = .2$$

$x_0 = 1$

$x_1 = X_0 + \Delta X = 1 + .2 = 1.2$

$+\Delta x$

$x_2 = X_0 + 2\Delta X = 1.2 + .2 = 1.4$

$x_3 = 1.6$

$x_4 = 1.8$

$x_5 = 2.0$

$$\overline{x_1} = \frac{x_0 + x_1}{2} = \frac{1 + 1.2}{2} = \frac{2.2}{2} = 1.1$$

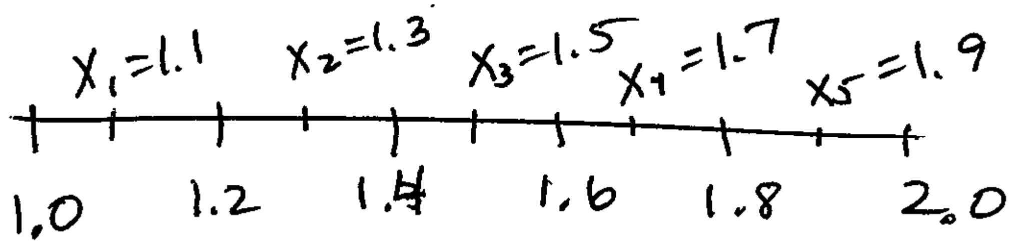
$+\Delta x = .2$

$$\overline{x_2} = \frac{x_1 + x_2}{2} = \frac{1.2 + 1.4}{2} = 1.3$$

$\overline{x_3} = 1.5$

$\overline{x_4} = 1.7$

$\overline{x_5} = 1.9$



$$\cancel{M_5}\ M_n = \Delta x \left[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n) \right]$$

$$= (.2)\left[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9) \right]$$

Note $f(x) = \frac{1}{x}$

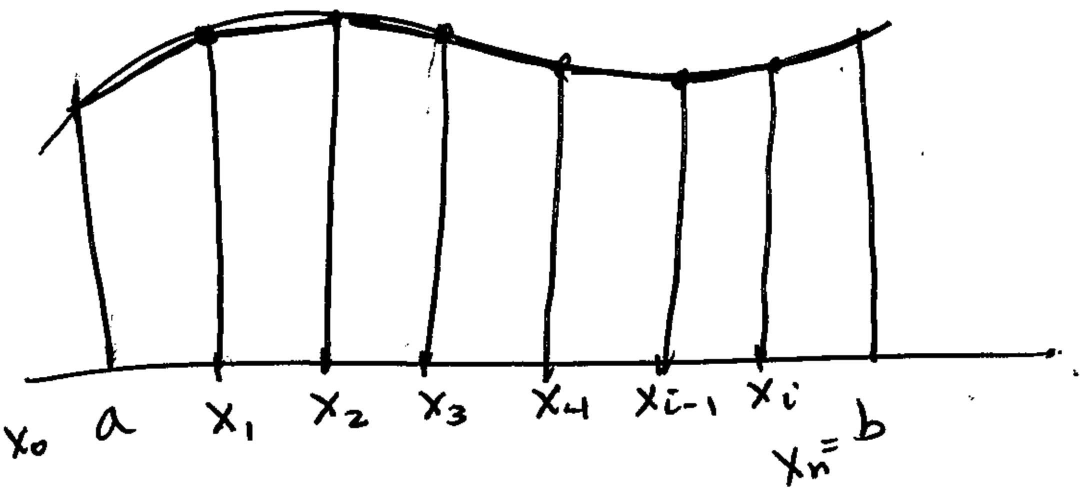
$$= (.2)\left[\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9} \right]$$

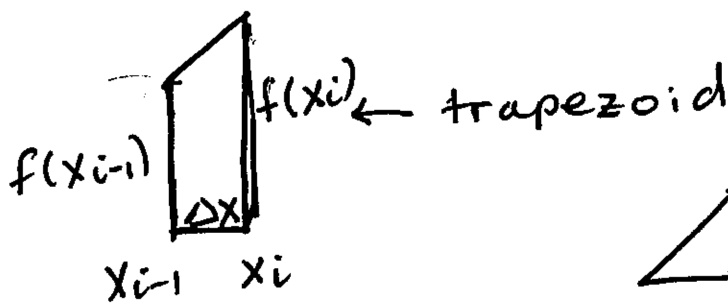
$$\approx 0.691908$$

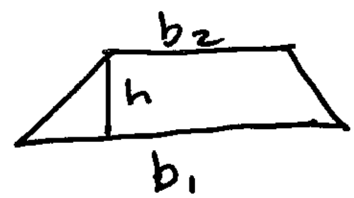
Note $\ln 2 \approx 0.693147\ldots$

The Trapezoid Rule

$$\text{Area} = \int_a^b f(x)\,dx$$







Area of ith trapezoid is

$$A_i = \frac{f(x_{i-1}) + f(x_i)}{2} \cdot \Delta x$$

Area of Trapezoid

$$A = \left(\frac{b_1 + b_2}{2}\right) h$$

Add up the A_i's

$$T_n = \frac{f(x_0) + f(x_1)}{2}\Delta x + \frac{f(x_1) + f(x_2)}{2}\Delta x + \cdots \frac{f(x_{n-1}) + f(x_n)}{2}\Delta x$$

$$T_n = \Delta x\left[\frac{f(x_0)}{2} + \left(\frac{f(x_1)}{2} + \frac{f(x_1)}{2}\right) + \left(\frac{f(x_2)}{2} + \frac{f(x_2)}{2}\right) + \cdots + \cdots \frac{f(x_{n-1}) + f(x_{n-1})}{2} + \frac{f(x_n)}{2}\right]$$

$$\boxed{T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots 2f(x_{n-1}) + f(x_n)\right]}$$

Trapezoid Rule

$$\Delta x = \frac{b-a}{n}, \quad x_i = a + i\Delta x$$

Example: Calculate T_5, $n=5$, for $\int_1^2 \frac{1}{x}\,dx$.

SOLUTION We have

$n=5$

$a=1$

$b=2$

$\Delta x = \frac{b-a}{n} = \frac{2-1}{5} = .2$

$f(x) = \frac{1}{x}$

$x_0 = 1$

$x_1 = x_0 + \Delta x = 1.2$ $\quad +\Delta x$

$x_2 = 1.4$

$x_3 = 1.6$

$x_4 = 1.8$

$x_5 = 2.0$

$$T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \cdots 2f(x_{n-1}) + f(x_n)\right]$$

$$= \frac{.2}{2}\left[f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)\right]$$

$f(x) = \frac{1}{x}$

$$= \frac{.2}{2}\left[\frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2.0}\right]$$

$$\approx 0.695635$$

Define the error as

$$T_n + E_T = \int_a^b f(x)\,dx$$

$$\boxed{E_T = \int_a^b f(x)\,dx - T_n}$$

$$E_T \approx 0.693147 - 0.695635$$
$$= -0.002488$$

While $E_M = \int_a^b f(x)\,dx - M_n$

error for midpoint rule

$$E_M \approx 0.001239$$

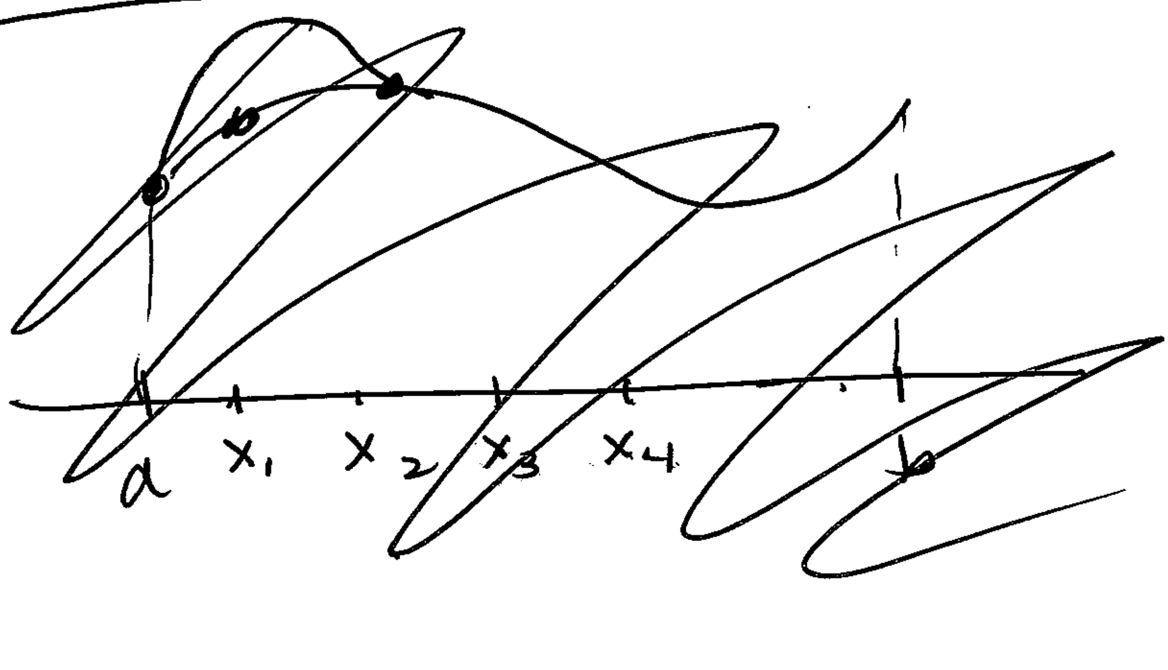
Simpson's Rule

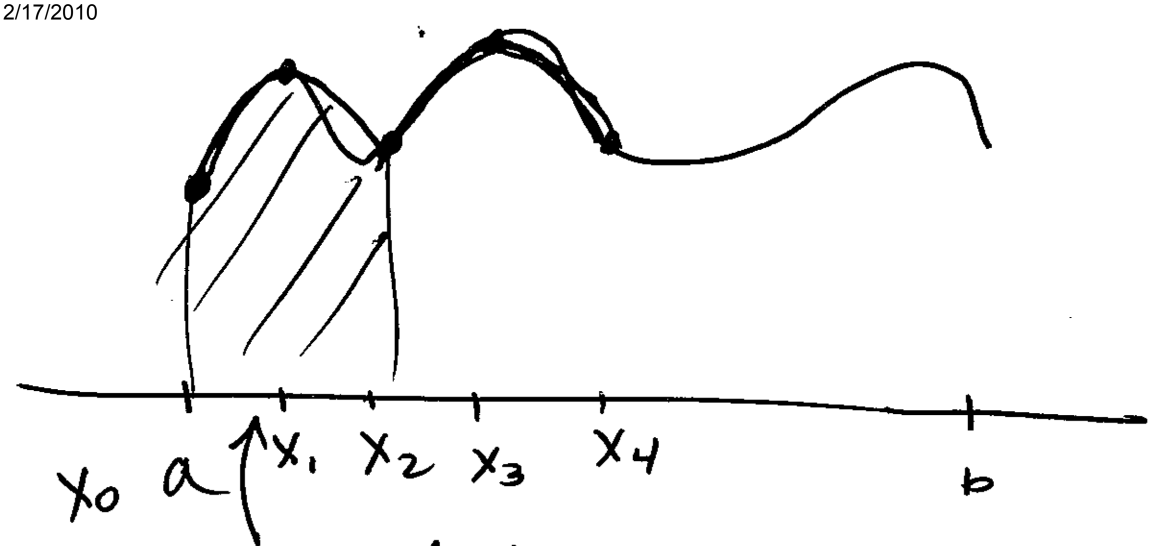
$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}\Big[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\Big]$$

where n is even

$$\Delta x = \frac{b-a}{n}$$

Explanation:





You find the area by integrating

$$\int_{x_0}^{x_2} (ax^2+bx+c)\,dx$$

what is a, b, c?

Given the three points, we can solve for a, b, c.

Example 4 Use Simpson's Rule with $n=10$ to approximate $\int_1^2 \frac{1}{x}\,dx$

SOLUTION

$n=10$

$f(x)=\frac{1}{x}$

$a=1$

$b=2$

$\Delta x = \frac{b-a}{n} = \frac{2-1}{10} = \frac{1}{10} = .1$

$x_0 = 1$ + Δx

$x_1 = 1.1$

$x_2 = 1.2$

$x_3 = 1.3$

$\vdots$

$x_9 = 1.9$

$x_{10} = 2.0$

FORMULA

$$S_n = \frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3) + \cdots 2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right]$$

$$= \frac{.1}{3}\left[f(1)+4f(1.1)+2f(1.2)+4f(1.3) +2f(1.\cancel{5}4)+4f(1.\cancel{6}5)+2f(1.6) + 4f(1.7)+2f(1.8)+4f(1.9)+f(2)\right]$$

$$= \frac{.1}{3}\left[\frac{1}{1}+\frac{4}{1.1}+\frac{2}{1.2}+\frac{4}{1.3}+\frac{2}{1.4}+\frac{4}{1.5}+\frac{2}{1.6} +\frac{4}{1.7}+\frac{2}{1.8}+\frac{4}{1.9}+\frac{1}{2}\right] \approx 0.693150$$

 ~~The Simpsons~~

- Simpson's Rule gives the best approximation.
- Midpoint Rule is next best.
- Trapezoid Rule comes in last place.

ERROR Bounds

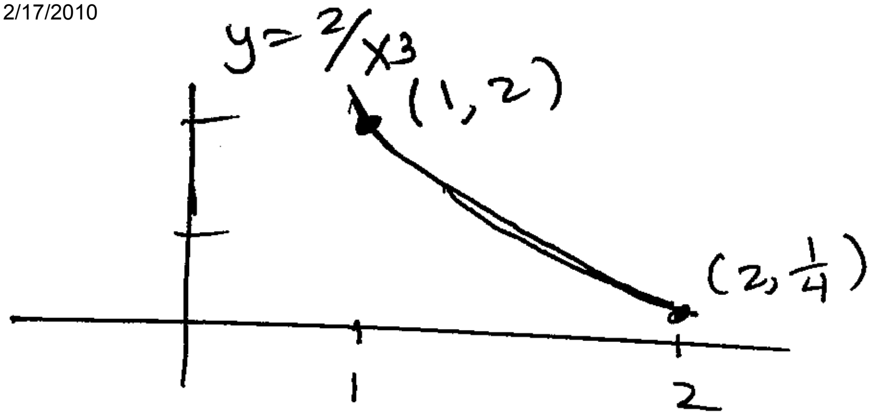
<u>Definition</u>

- Simpson's Rule ERROR $E_S = \int_a^b f(x)\,dx - S_n$
- midpt rule approx $E_M = \int_a^b f(x)\,dx - M_n$

$E_T = \int_a^b f(x)\,dx - T_n$

Trap. ERROR

How big should we choose n to be?



x	$2/x^3$
1	$2/1^3 = 2$
2	$2/2^3 = \frac{1}{4}$

$$\text{Max of } |f''(x)| = 2$$

$$\underline{\underline{\text{Let } K = 2}}$$

$K = 2$

$a = 1$

$b = 2$

$n = ?$ ← find

Max error $= 0.0001 = 10^{-4} = \frac{1}{10{,}000} = \frac{1}{10^4}$

- Trap. Rule

$$|E_T| \le \frac{K(b-a)^3}{12n^2} \le 0.0001$$

$$\frac{2(2-1)^3}{12n^2} \le \frac{1}{10^4} \quad \text{solve for } n.$$

$$\frac{\cancel{2}}{{}_6\cancel{12}n^2} \le \frac{1}{10^4}$$

$$\frac{1}{6n^2} \le \frac{1}{10^4}$$

$$10^4 \le 6n^2$$

$$\frac{10^4}{6} \le n^2$$

$$\sqrt{\frac{10^4}{6}} \le n$$

$$40.82 \le n$$

$$n = 41$$

Error Bounds

Suppose $|f''(x)| \le K$ for $a \le x \le b$. If E_T and E_M are the errors for the Trapezoid and Midpoint rules respectively, then

$$|E_T| \le \frac{K(b-a)^3}{12n^2} \quad \text{and} \quad |E_M| \le \frac{K(b-a)^3}{24n^2}$$

Example: How large should we take n in order to guarantee that the Trapezoid Rule and Midpoint rule ~~for~~ approximations for $\int_1^2 \frac{1}{x}\,dx$ are accurate within 0.0001?

SOLUTION

Find K.

$f(x) = \frac{1}{x} = x^{-1}$

$f'(x) = -1 \cdot x^{-2} = -x^{-2}$

$f''(x) = -1 \cdot (-2)x^{-3} = \frac{2}{x^3}, \quad 1 \le x \le 2$

What is the max of $|f''(x)| = \left|\frac{2}{x^3}\right|$ on $[1,2]$?

§7.7 HW # 7-~~23~~ 21 odd

~~Tomm~~ Tomorrow

Quiz ~~§7.2, 7.3~~

§7.3, 7.4