§7.8 Continued

Math 185
~~Th~~ Tues 3/02

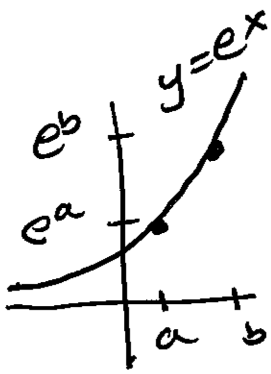
A Comparison Test for Improper Integrals

<u>Example</u>: Show that

$$\int_0^\infty e^{-x^2}\,dx \quad \text{is convergent.}$$

$$= \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx$$

Aside



if $a \le b$ then $e^a \le e^b$

$x \le x^2$ for $1 \le x$

multiply through by -1

$-x \geq -x^2$ ← mult. by neg, change direction of ineq.

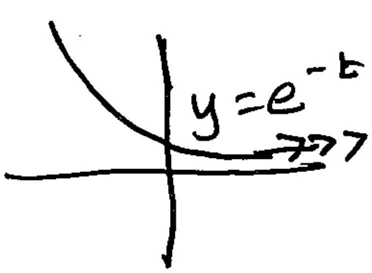
$-x^2 \le -x$

$e^{-x^2} \le e^{-x}$

• $\displaystyle\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty} \int_1^t e^{-x}\,dx$

$\displaystyle = \lim_{t\to\infty} \left[-e^{-x}\right]_1^t$

$\displaystyle = \lim_{t\to\infty} \left(\underset{\text{I}}{-e^{-t}}\right) - \left(-e^{-1}\right) = \frac{1}{e}$



I $\displaystyle\lim_{t\to\infty} -e^{-t} = 0$

$\displaystyle\int_1^\infty e^{-x}\,dx$ is <u>convergent</u>

1

Example: show that

$$\int_1^{\infty} \frac{1+e^{-x}}{x}\,dx \text{ is divergent.}$$

SOLUTION

- $\dfrac{1}{x} \le \dfrac{1+e^{-x}}{x}$ $\qquad 1 \le x$

$$\boxed{\begin{array}{l} 1 = 1 \\ 1 \le 1 + \underbrace{e^{-x}}_{\text{always pos}} \end{array}}$$

- $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x}\,dx$

$$= \lim_{t\to\infty} \ln|x| \Big]_1^t$$

$$= \lim_{t\to\infty} \left(\ln|t| - \ln|1|\right)$$

$$= \infty \quad \text{divergent.}$$

Therefore by Comp. Test

$$\int_1^{\infty} \frac{1+e^{-x}}{x}\,dx \text{ is divergent.}$$

We have

$$0 \le e^{-x^2} \le e^{-x} \quad \text{for } 1 \le x$$

and $\int_1^{\infty} e^{-x}\,dx$ is convergent

Therefore $\int_1^{\infty} e^{-x^2}\,dx$ is convergent by comparison Theorem

<u>Comparison Theorem</u> Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$

(a) If $\int_a^{\infty} f(x)\,dx$ is convergent, then $\int_a^{\infty} g(x)$ is convergent.

(b) If $\int_a^{\infty} g(x)\,dx$ is divergent, then $\int_a^{\infty} f(x)\,dx$ is divergent.