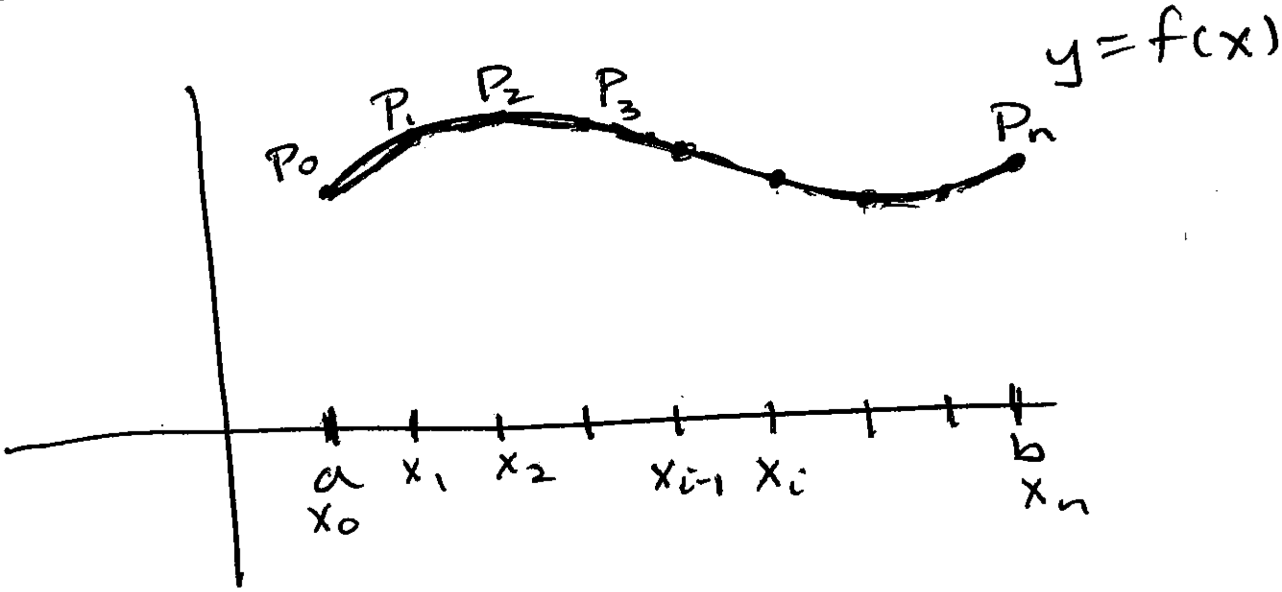
§ 8.1 Arc Length

§ 8.1 HW # 1-18

Math 185
Weds. 3/03/2010



Goal: To find the length of a curve given by a ~~continuous~~ differentiable function on a closed interval.

- We make a partition of [a, b].

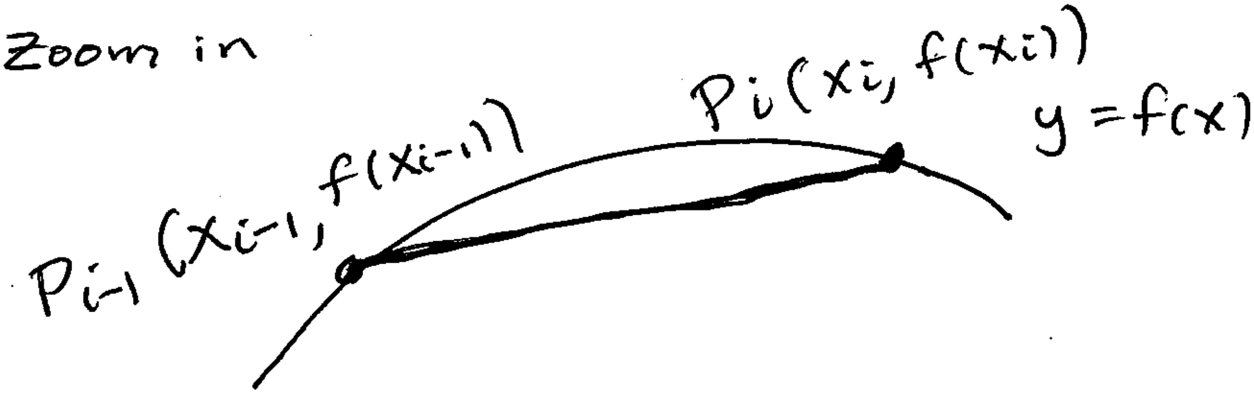
$x_0, x_1, x_2, \ldots x_n$.

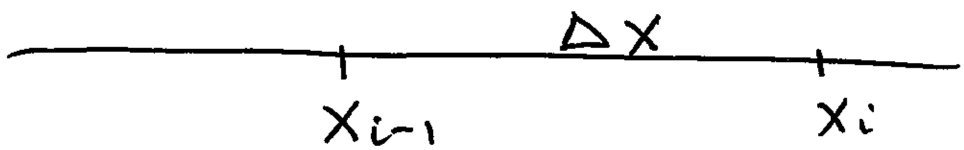
we connect points

$(x_0, f(x_0)), (x_1, f(x_1)) \ldots (x_n, f(x_n))$

by lines.

Zoom in





The length of ~~$P_{i-1}P_i$~~
~~the interval from P_i-~~
the line segment from P_{i-1} to P_i, denoted $|P_{i-1}P_i|$

is

$$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$$

$$= \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$

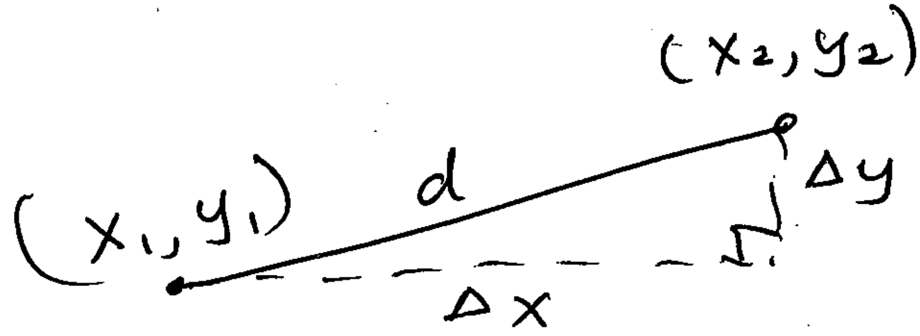
A litte bit more advanced:

Mean Value Theorem:

If f is differentiable on (a,b), continuous on $[a,b]$, then there exists c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

Aside



$$d^2 = (\Delta x)^2 + (\Delta y)^2$$

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

If we apply the Mean Value Theorem to f on $[x_{i-1}, x_i]$ we ~~say~~ can conclude that there is a number x_i^* in (x_{i-1}, x_i) such that

$$f'(x_i^*) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

i.e.

$$f'(x_i^*) = \frac{\Delta y_i}{\Delta x}$$

We have $f'(x_i^*)\Delta x = \Delta y_i$

Therefore

$$|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + (\underbrace{f'(x_i^*)\Delta x}_{\Delta y_i})^2}$$

$$= \sqrt{(1 + (f'(x_i^*))^2)(\Delta x)^2}$$

$$= \sqrt{1 + (f'(x_i^*))^2} \cdot \Delta x$$

We sum all the line segments

$$\sum_{i=1}^{n} |P_{i-1}P_i| = \sum_{i=1}^{n} \sqrt{1+(f'(x_i^*))^2}\,\Delta x$$

We take the limit as $n \to \infty$

We define arc length, L, as

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1+(f'(x_i^*))^2}\,\Delta x$$

$$L = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

where $f'(x)$ is continuous.

We can also write

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Example: Find the length of the arc of the semicubical parabola $y^2 = x^3$ between the points $(1,1)$ and $(4,8)$.

SOLUTION

$$y^2 = x^3 \qquad a = 1$$

$$y = x^{3/2} \qquad b = 4$$

$$\frac{dy}{dx} = \frac{3}{2} x^{1/2}$$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_1^4 \sqrt{1 + \left(\tfrac{3}{2}x^{1/2}\right)^2}\, dx$$

$$= \int_1^4 \sqrt{1 + \tfrac{9}{4}x}\, dx \qquad u = 1 + \tfrac{9}{4}x$$

$$du = \tfrac{9}{4}\, dx$$

$$= \int \sqrt{u} \cdot \tfrac{4}{9}\, du \qquad \tfrac{4}{9}\, du = dx$$

$$= \frac{4}{9}\int u^{1/2}\, du$$

$$= \frac{4}{9}\, \frac{u^{3/2}}{3/2} + \cancel{C} = \left(\frac{2}{3}\right)\left(\frac{4}{9}\right)\left(1 + \frac{9}{4}x\right)^{3/2}\Bigg]_1^4$$

$$= \frac{8}{27}\left(1 + \frac{9}{4}\cdot 4\right)^{3/2} - \frac{8}{27}\left(\frac{4}{4} + \frac{9}{4}\cdot 1\right)^{3/2}$$

$$= \frac{8}{27}\left[(10)^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right]$$

$$= \frac{8}{27}\left[10\sqrt{10} - \frac{13}{4}\sqrt{\frac{13}{4}}\right]$$

$$\boxed{\begin{aligned} x^{3/2} &= x \cdot x^{1/2} \\ &= x\sqrt{x} \end{aligned}}$$

$$= \frac{8}{27}\left[10\sqrt{10} - \frac{13}{4}\cdot\frac{\sqrt{13}}{2}\right]$$

$$= \frac{\cancel{8}}{27}\cdot\frac{1}{\cancel{8}}\left[80\sqrt{10} - 13\sqrt{13}\right]$$

$$= \frac{1}{27}\left[80\sqrt{10} - 13\sqrt{13}\right]$$