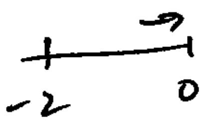
§7.8 #31 $\displaystyle\int_{-2}^{3} \frac{1}{x^4}\,dx$

$\frac{1}{x^4}$ discont. at $x=0$

$$= \overset{\text{I}}{\int_{-2}^{0} \frac{1}{x^4}\,dx} + \overset{\text{II}}{\int_{0}^{3} \frac{1}{x^4}\,dx}$$

$$\int \frac{1}{x^4}\,dx = \int x^{-4}\,dx = \frac{x^{-3}}{-3} = \frac{-1}{3x^3}$$

I $\displaystyle\int_{-2}^{0} \frac{1}{x^4}\,dx = \lim_{t\to 0^-} \int_{-2}^{t} \frac{1}{x^4}\,dx = \lim_{t\to 0^-} \left[\frac{-1}{3x^3}\right]_{-2}^{t}$



$$= -\frac{1}{3} \lim_{t\to 0^-} \left[\frac{1}{t^3} - \frac{1}{(-2)^3}\right]$$

$\displaystyle\lim_{t\to 0^-} \frac{1}{t^3} = -\infty$ Divergent.

Test number $t = -.001$

$y = \dfrac{1}{(-.001)^3} = \text{big}$

If we have
a limit where
after direct
substitution we get $\frac{1}{0}$
(or a nonzero number over 0)
then the limit is divergent.

§8.1 #9 $\quad y = \frac{x^5}{6} + \frac{1}{10x^3}, \quad 1 \le x \le 2$

Find the length of the curve.

$$L = \int ds$$

$$ds = \sqrt{1+(y')^2}\,dx$$

or

$$ds = \sqrt{1+(x')^2}\,dy$$

Find y': $\quad y = \frac{x^5}{6} + \frac{x^{-3}}{10}$

$$y' = 5\frac{x^4}{6} + \frac{-3x^{-4}}{10}$$

$$L = \int_1^2 \sqrt{1 + \left(\frac{5}{6}x^4 - \frac{3}{10x^4}\right)^2}\, dx$$

$$= \int_1^2 \sqrt{1 + \frac{25}{36}x^8 - \cancel{2}\left(\frac{5}{\cancel{6}_2}\cancel{x^4}\right)\left(\frac{\cancel{3}}{\cancel{10}\cancel{x^4}}\right) + \frac{9}{100x^8}}\, dx$$

$$= \int_1^2 \sqrt{1 + \frac{25}{36}x^8 - \frac{1}{2} + \frac{9}{100x^8}}\, dx$$

$$= \int_1^2 \sqrt{\frac{25}{36}x^8 + \frac{1}{2} + \frac{9}{100x^8}}\, dx$$

$$= \int_1^2 \sqrt{\left(\frac{5}{6}x^4 + \frac{3}{10x^4}\right)^2}\, dx$$

$$= \int_1^2 \left(\frac{5}{6}x^4 + \frac{3}{10}x^{-4}\right) dx$$

$$= \left[\frac{\cancel{5}}{6} \cdot \frac{x^5}{\cancel{5}} + \frac{\cancel{3}}{10} \frac{x^{-3}}{-\cancel{3}} \right]_1^2$$

$$= \left[\frac{2^5}{6} - \frac{2^{-3}}{10} \right] - \left[\frac{1^5}{6} - \frac{1^{-3}}{10} \right]$$

$$= \frac{32}{6} - \frac{1}{10} \cdot \frac{1}{8} - \frac{1}{6} + \frac{1}{10} \cdot \frac{8}{8}$$

$$= \frac{31}{6} + \frac{7}{80} = \frac{31 \cdot 40}{6 \cdot 40} + \frac{7 \cdot 3}{80 \cdot 3}$$

$$= \frac{1261}{240}$$

$$\begin{array}{r} 31 \\ 40 \\ \hline 1240 \\ +21 \end{array}$$

§8.1 #4

$y = xe^{-x^2}, \quad 0 \le x \le 1$

Find a formula for L.

$$L = \int ds = \int_a^b \sqrt{1+(y')^2}\,dx$$

$$y' = (x)'(e^{-x^2}) + (x)(e^{-x^2})'$$

$$y' = 1 \cdot e^{-x^2} + x(e^{-x^2} \cdot (-2x))$$

$$y' = e^{-x^2}(1-2x^2)$$

$$L = \int_0^1 \sqrt{1+[e^{-x^2}(1-2x^2)]^2}\,dx$$

$$= \int_0^1 \sqrt{1+(e^{-x^2})^2(1-2x^2)^2}\,dx$$

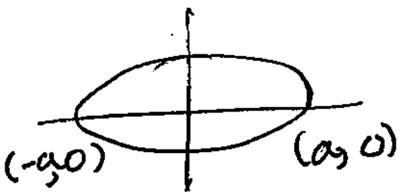
$$= \int_0^1 \sqrt{1+e^{-2x^2}(1-2x^2)^2}\,dx$$

§8.1 #6 $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Find y'.

~~Let's use implicit diff!~~

~~$\frac{2x}{a^2} +$~~



When $y=0$

$\frac{x^2}{a^2} + \frac{0^2}{b^2} = 1$

$x^2 = a^2$

$x = \pm a$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y^2 = b^2 - \frac{b^2}{a^2}x^2$$

$$y = \sqrt{b^2 - \frac{b^2}{a^2}x^2} \quad \leftarrow \text{top of ellips}$$

$$y = b\left(1 - \frac{x^2}{a^2}\right)^{1/2}$$

$$y' = b \cdot \frac{1}{\cancel{2}}\left(1 - \frac{x^2}{a^2}\right)^{-1/2} \cdot \left(-\frac{\cancel{2}x}{a^2}\right)$$

$$y' = \frac{-bx}{a^2\sqrt{1 - \frac{x^2}{a^2}}}$$

~~Let's~~ ~~$(y')^2$~~

mult by top & bottom
2 to get

$$L = 2\int_{-a}^{a} \sqrt{1 + \left(\frac{-bx}{a^2\sqrt{1-\frac{x^2}{a^2}}}\right)^2}\, dx$$

$$= \frac{\pi}{18}\left[\frac{u^{3/2}}{3/2}\right]_1^{145}$$

$$= \frac{\cancel{2}}{3}\,\frac{\pi}{\cancel{18}\,9}\left[(145)^{3/2} - (1)^{3/2}\right]$$

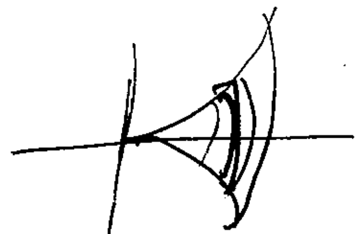
$$= \frac{\pi}{27}\left[145\sqrt{145} - 1\right]$$

~~§8.2 #5~~ Find the area of the surface found by rotating the curve about the x-axis ~~given~~ axis.

§8.2 #5 $\quad y = x^3, \quad 0 \le x \le 2$

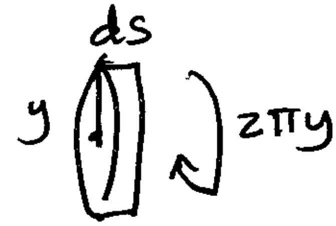
about x-axis

$$S = \int 2\pi y \, ds$$



$$ds = \sqrt{1+(y')^2}\,dx$$

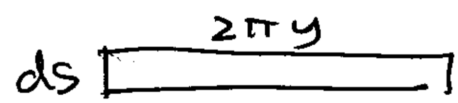
$$ds = \sqrt{1+(x')^2}\,dy$$



Let's find y'

$y = x^3$

$y' = 3x^2$



$$ds^2 = dx^2 + dy^2$$

$$S = \int 2\pi y \sqrt{1+(y')^2}\,dx$$

$$= \int_0^2 2\pi x^3 \sqrt{1+(3x^2)^2}\,dx$$

$$= \int_0^2 2\pi x^3 \sqrt{1+9x^4}\,dx$$

$\begin{array}{r} ^5 16 \\ 9 \\ \hline 144 \end{array}$

$u = 1 + 9x^4$

$du = 36x^3\,dx$

$\frac{1}{36}du = x^3\,dx$

when $x = 0$
$u = 1 + 9 \cdot 0^4 =$
when $x = 2$
$u = 1 + 9(2)^4$
$= 1 + 9 \cdot 16$
$= 145$

$$= \int_1^{145} 2\pi \cdot \frac{1}{36}(u)^{1/2}\,du$$

Surface Area

§8.2 #11 about the x-axis

$$x = \frac{1}{3}(y^2+2)^{3/2}, \quad 1 \le y \le 2$$

$$\boxed{S = \int 2\pi y \, ds}$$

$$ds = \sqrt{1+(x')^2}\,dy$$

or

$$ds = \sqrt{1+(y')^2}\,dx$$

Find x' ~~(that~~ (that is $\frac{dx}{dx}$)

$$x' = \frac{1}{3} \cdot \frac{3}{2}(y^2+2)^{1/2} \cdot 2y$$

$$x' = y(y^2+2)^{1/2}$$

$$\boxed{ds^2 = dx^2 + dy^2}$$

$$ds = \sqrt{dx^2+dy^2}$$

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2} \cdot d$$

$$S = \int 2\pi y \, ds$$

$$= \int_c^d 2\pi y \sqrt{1+(x')^2}\,dy$$

$$= \int_1^2 2\pi y \sqrt{1+\left(y(y^2+2)^{1/2}\right)^2}\,dy$$

$$= \int_1^2 2\pi y \sqrt{1+y^2(y^2+2)}\,dy$$

$$= \int_1^2 2\pi y \sqrt{1+y^4+2y^2}\,dy$$

$$= \int_1^2 2\pi y \sqrt{(y^2+1)^2}\,dy$$

$$= \int_1^2 2\pi y (y^2+1)\,dy$$

$$= \int_1^2 2\pi\,(y^3 + y)\,dy$$

$$= 2\pi\left[\frac{y^4}{4} + \frac{y^2}{2}\right]_1^2$$

$$= 2\pi\left[\frac{2^4}{4} + \frac{2^2}{2}\right] - 2\pi\left[\frac{1^4}{4} + \frac{1^2}{2}\right]$$

$$= 2\pi\left[4 + 2\right] - 2\pi\left[\frac{1}{4} + \frac{1}{2}\right]$$

$$= 2\pi(6) - \cancel{2}\pi\left(\frac{3}{\cancel{4}_2}\right)$$

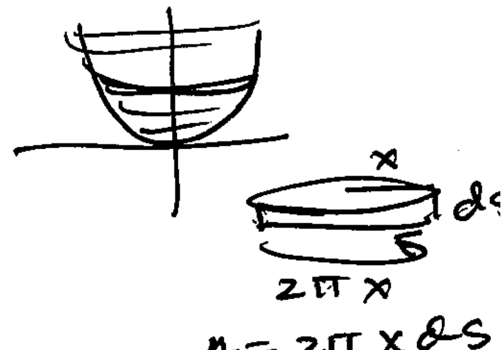
$$= 12\pi - \frac{3}{2}\pi = \frac{24}{2}\pi - \frac{3}{2}\pi = \frac{21}{2}\pi$$

§8.2 #15 Surface Area about the y-axis.

$$x=\sqrt{a^2-y^2}, \quad 0 \le y \le \frac{a}{2}$$

$$S=\int 2\pi x \, ds$$

$$ds=\sqrt{1+(x')^2}\, dy$$



Find x'

$$x=(a^2-y^2)^{1/2}$$

$$x'=\frac{1}{2}(a^2-y^2)^{-1/2}(-2y)$$

$$x'=\frac{-y}{\sqrt{a^2-y^2}}$$

$$S=\int 2\pi\, x\sqrt{1+(x')^2}\, dy$$

$$=\int 2\pi\sqrt{a^2-y^2}\sqrt{1+\left(\frac{-y}{\sqrt{a^2-y^2}}\right)^2}\, dy$$

$$=\int_0^{a/2} 2\pi\sqrt{a^2-y^2}\sqrt{1+\left(\frac{-y}{\sqrt{a^2-y^2}}\right)^2}\, dy$$

$$=2\pi\int_0^{a/2}\sqrt{a^2-y^2}\sqrt{1+\frac{y^2}{a^2-y^2}}\, dy$$

$$=2\pi\int_0^{a/2}\sqrt{a^2-y^2}\sqrt{\frac{a^2-y^2}{a^2-y^2}+\frac{y^2}{a^2-y^2}}\, dy$$

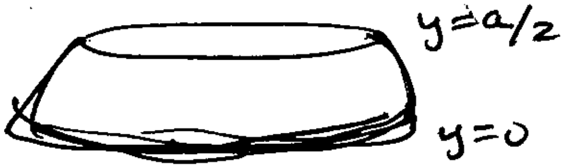
$$= 2\pi \int_0^{a/2} \sqrt{a^2-y^2}\,\sqrt{\frac{a^2}{a^2-y^2}}\,dy$$

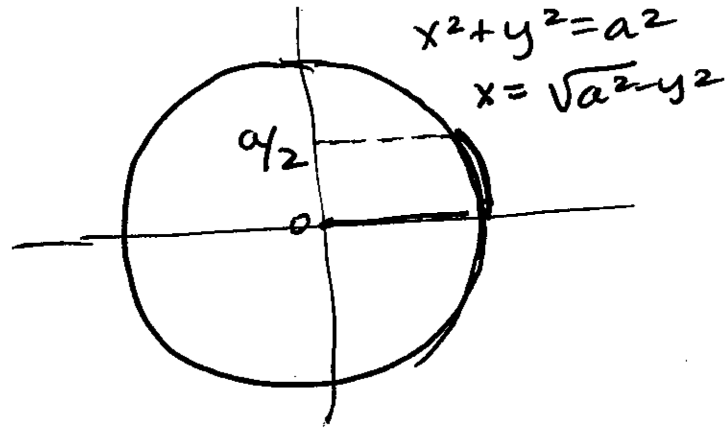
$$= 2\pi \int_0^{a/2} \cancel{\sqrt{a^2-y^2}}\,\frac{a}{\cancel{\sqrt{a^2-y^2}}}\,dy$$

$$= 2\pi a \int_0^{a/2} dy = 2\pi a\left[y\right]_0^{a/2}$$

$$= 2\pi a\left[a/2 - 0\right]$$

$$= \pi a^2$$



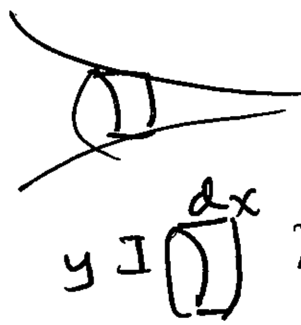


§8.2 #25 Gabriel's Horn

$$\mathcal{R} = \{(x,y) \mid x \geq 1,\ 0 \leq y \leq 1/x\}$$

is rotated about the x-axis.

The volume of the resulting solid is finite.



$$V = \int_a^b \pi (f(x))^2\, dx$$

Area $\pi r^2 = \pi y^2$

$$= \int_1^\infty \pi \left(\frac{1}{x}\right)^2 dx$$

$$= \pi \int_1^\infty \frac{1}{x^2}\, dx$$

Volume $\pi y^2\, dx$

$\int_1^\infty \frac{1}{x^p}\, dx$ is convergent if $p > 1$

so this volume is finite.
(here $p = 2$)

Show that the surface ~~area~~ is infinite

$y = \frac{1}{x}, \; y' = \frac{-1}{x^2}$

$$S = \int 2\pi y \, ds$$

$$= \int_a^b 2\pi y \sqrt{1 + (y')^2} \, dx$$

$$= \int_1^\infty 2\pi \left(\frac{1}{x}\right) \sqrt{1 + \left(\frac{-1}{x^2}\right)^2} \, dx$$

$$= 2\pi \int_1^\infty \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} \, dx$$

$$= 2\pi \int_1^\infty \frac{1}{x} \sqrt{\frac{x^4}{x^4} + \frac{1}{x^4}} \, dx$$

$$= 2\pi \int_1^\infty \frac{1}{x} \sqrt{\frac{x^4+1}{x^4}} \, dx$$

$$= 2\pi \int_1^\infty \frac{1}{x} \cdot \frac{1}{x^2} \sqrt{x^4+1} \, dx$$

$$= 2\pi \int_1^\infty \frac{1}{x^3} \sqrt{x^4+1} \, dx$$

$$0 < \frac{1}{x} = \frac{x^2}{x^3} = \frac{\sqrt{x^4}}{x^3} \le \frac{\sqrt{x^4+1}}{x^3}$$

$\int_1^\infty \frac{1}{x} dx$ divergent

$\therefore$ Divergent by comp. Test, ~~comp. Th~~
so infinite surface area

(13)