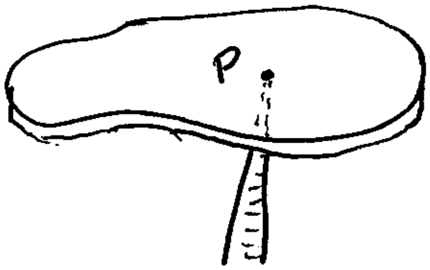
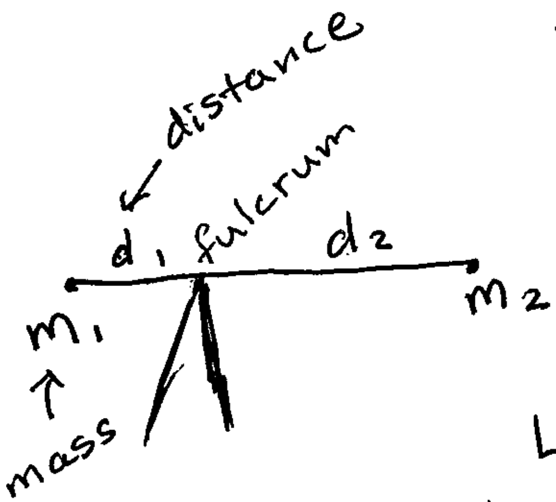
§8.3 Applications to Physics and Engineering

HW §8.3 # 23 - 35 odd



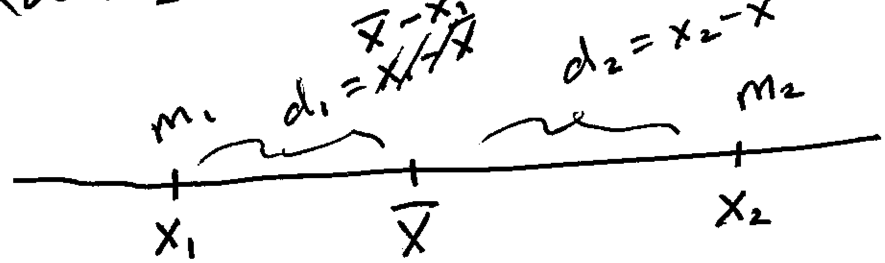
Objective: To find the point P on which a thin plate of any shape balances horizontally. Such a point is called the center of mass



The rod will balance if $m_1 d_1 = m_2 d_2$

This is called the Law of Lever discovered by ~~Archmi~~ Archimedes.

Suppose the masses lie along the x-axis, with m_1 at x_1 and m_2 at x_2 and center of mass $\bar{x}$.



$$m_1 d_1 = m_2 d_2$$

$$m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$$

$$m_1 \bar{x} - m_1 x_1 = m_2 x_2 - m_2 \bar{x}$$

$$m_1 \bar{x} + m_2 \bar{x} = m_1 x_1 + m_2 x_2$$

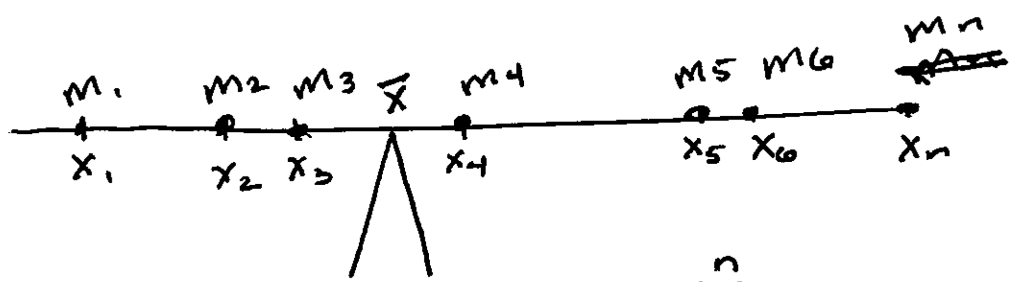
$$\cancel{\bar{x} = m_1 x + m_2 x}$$

$$\bar{x}(m_1 + m_2) = m_1 x_1 + m_2 x_2$$

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

The numbers $m_1 x_1$ and $m_2 x_2$ are called the moments of the masses m_1 and m_2.

The total mass is $m = m_1 + m_2$. The above equation says that the center of mass $\bar{x}$ is the sum of the moments of the masses divided by the total mass.



We can generalize $\bar{x} = \dfrac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \dfrac{\sum_{i=1}^{n} m_i x_i}{m}$

The sum of the moments is

$$M = \sum_{i=1}^{n} m_i x_i$$

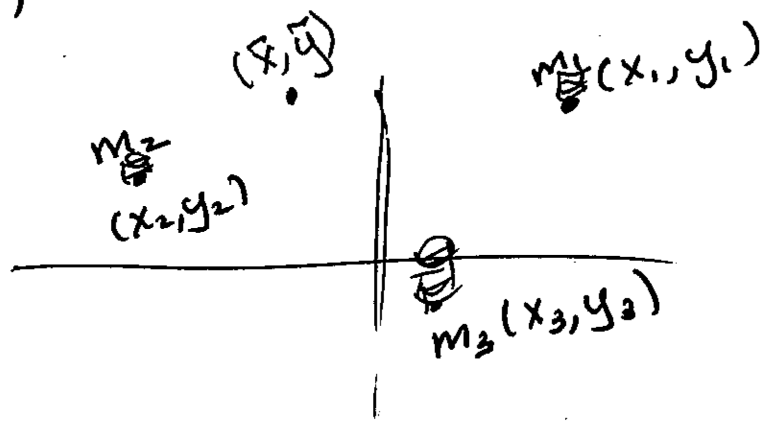
is called the <u>moment of the system about the origin</u>.

and we have total mass

$$m = \sum_{i=1}^{n} m_i .$$

So $\bar{x} = \frac{M}{m}$ that is $m\bar{x} = M$

We now suppose masses $m_1, m_2, \ldots, m_n$ are located in the plane at points $(x_1, y_1), (x_2, y_2), \ldots (x_n, y_n)$



We define the <u>moment of the system about the y-axis</u> to be

$$M_y = \sum_{i=1}^{n} m_i x_i$$

and the moment about the x-axis is

$$M_x = \sum_{i=1}^{n} m_i y_i$$

M_y measures the tendency ~~to~~ of the system to rotate about the y-axis; M_x measures the tendency of the system to rotate about the x-axis.

We define the center of mass $(\bar{x}, \bar{y})$ by $\bar{x} = \dfrac{M_y}{m}$, $\bar{y} = \dfrac{M_x}{m}$

$$m = \sum_{i=1}^{n} m_i .$$

The center of mass $(\bar{x}, \bar{y})$ is the point where a single particle of mass m would have the same moments as the system.

EXAMPLE: Find the moments and center of mass of the system of objects that have masses 3, 4, and 8 at the points $(-1,1)$, $(2,-1)$, and $(3,2)$.

~~Karaoke~~

~~Croquette~~

SOLUTION

$m_1 = 3$, $x_1 = -1$, $y_1 = 1$

$m_2 = 4$, $x_2 = 2$, $y_2 = -1$

$m_3 = 8$, $x_3 = 3$, $y_3 = 2$

total mass $\quad m = \sum_{i=1}^{3} m_i = 3 + 4 + 8 = 15$

moments $\quad M_y = \sum_{i=1}^{3} m_i x_i = (3)(-1) + (4)(2) + (8)(3) = -3 + 8 + 24 = 29$

$M_x = \sum_{i=1}^{3} m_i y_i = (3)(1) + (4)(-1) + (8)(2) = 3 - 4 + 16 = 15$

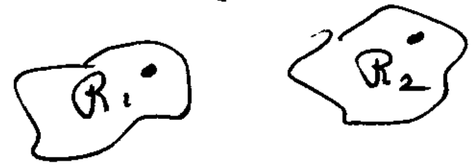
• center of mass $\bar{x} = \frac{M_y}{m} = \frac{29}{15}$

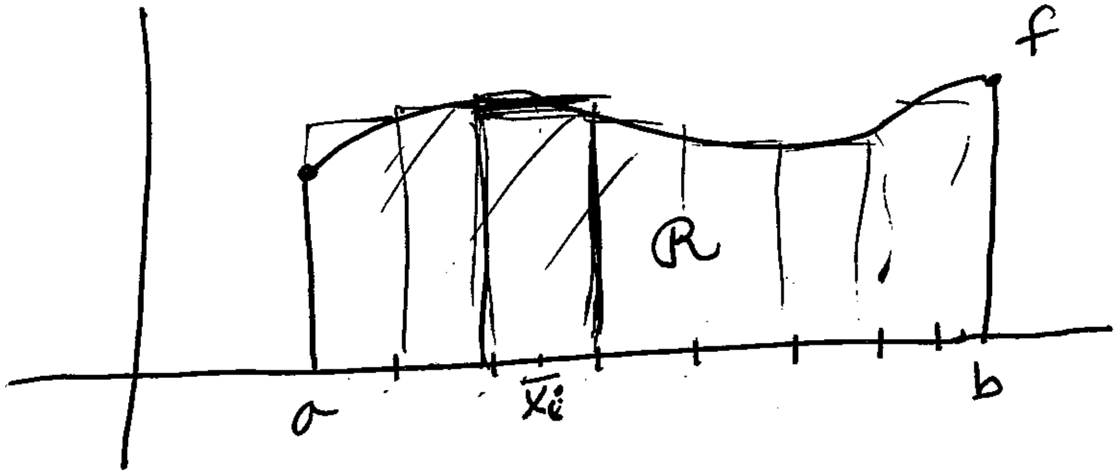
$$\bar{y} = \frac{M_x}{m} = \frac{15}{15} = 1$$

We now find the center of mass of a flat plate, called a lamina, with uniform density ρ (rho) that occupies a region R of the plane. The center of mass is called the centroid of R.

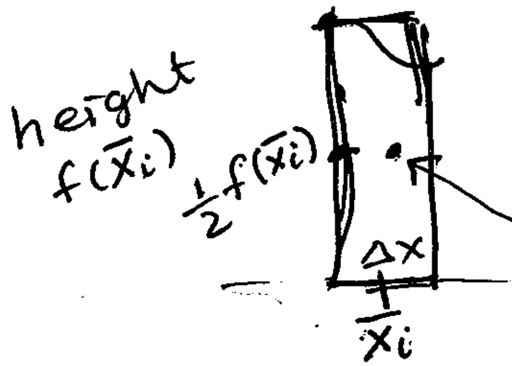
We assume these principles of physics.

- The symmetry principle: If R is symmetric about a line l, then the centroid of R lies on l.
- If the entire mass of a region is concentrated at the center of mass, then its moments remain unchanged.
- The moments of the union of two non-overlapping regions is the sum of the moments of the individual regions.





We partition $[a,b]$ into n subintervals with midpoints $\bar{x}_i$.



The ith rectangle

The center is given by $(\bar{x}_i, \frac{1}{2}f(\bar{x}_i))$

The area of the ith rectangle is $f(\bar{x}_i)\,\Delta x$

$\uparrow$ height $\uparrow$ base

Density is mass divided area.

$$\rho = \frac{\text{mass}}{\text{area}} = \frac{\text{mass}}{f(\bar{x}_i)\,\Delta x}$$

$$\text{mass} = \rho f(\bar{x}_i)\,\Delta x$$

The moment of the ith rectangle is ~~then~~ the product of the mass and the distance c_i to the y-axis.

$$M_y(R_i) = [\rho f(\bar{x}_i)\,\Delta x]\,\bar{x}_i = \rho\,\bar{x}_i f(\bar{x}_i)\,\Delta x$$

$\uparrow$ ith rectangle

We then add up all of the moments of the n rectangles to get the moment about the y-axis

$$\sum_{i=1}^{n} M_y(R_i) = \sum_{i=1}^{n} \rho \bar{x}_i f(\bar{x}_i) \Delta x$$

We let $n \to \infty$ and ~~is~~ define

$$M_y = \lim_{n \to \infty} \sum_{i=1}^{n} \rho \bar{x}_i f(\bar{x}_i) \Delta x$$

$$M_y = \rho \int_a^b x f(x)\, dx$$

We define the moment about the x-axis

$$M_x = \rho \int_a^b \frac{1}{2} [f(x)]^2 dx$$

The center of mass is $m\bar{x} = M_y$ and $m\bar{y} = M_x$.

~~We find the mass~~

The mass is the product of the density and the total area.

$$m = \rho A = \rho \int_a^b f(x) dx$$

We have

$$\bar{x} = \frac{M_y}{m} = \frac{\cancel{\rho}\int_a^b x f(x) dx}{\cancel{\rho}\int_a^b f(x) dx} = \frac{\int_a^b x f(x) dx}{\int_a^b f(x) dx}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\rho \int_a^b \frac{1}{2}[f(x)]^2 dx}{\rho \int_a^b f(x)\, dx} = \frac{\int_a^b \frac{1}{2}[f(x)]^2 dx}{\int_a^b f(x) dx}$$

$$\bar{x} = \frac{1}{A}\int_a^b x f(x)\,dx, \quad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\,dx$$

where $A = \int_a^b f(x)\,dx$ is the area.