Buff #7

Math 185

3|01

$$\int \frac{3x+5}{x^2+4x+13}$$

$$\int \frac{3x+5}{(x^2+4x+4)-4+13}dx = \int \frac{3x+5}{(x+2)^2+9}dx$$

$u = x+2, \quad x = u-2$

$du = dx$

$$= \int \frac{3(u-2)+5}{u^2+9}du = \int \frac{3u-6+5}{u^2+9}du$$

$$= \int \frac{3u-1}{u^2+9}du = \underset{\text{I}}{3\int \frac{u}{u^2+9}du} - \underset{\text{II}}{\int \frac{1}{u^2+9}du}$$

I $\quad 3\int \frac{u}{u^2+9}du$

$w = u^2+9$

$dw = 2u\,du$

$\frac{1}{2}dw = u\,du$

$$3 \cdot \frac{1}{2}\int \frac{1}{w}dw = \frac{3}{2}\ln|w| + C_1$$

$$= \frac{3}{2}\ln|u^2+9| + C_1$$

$$= \frac{3}{2}\ln|x^2+4x+13| + C_1$$

II $\quad -\int \frac{1}{u^2+9}du = -\frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right) + C_2$

$$= -\frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + C_2$$

$$= \frac{3}{2}\ln|x^2+4x+13| - \frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + C$$

0

§7.8 Improper Integrals

Hw §7.8 #1–49 odd

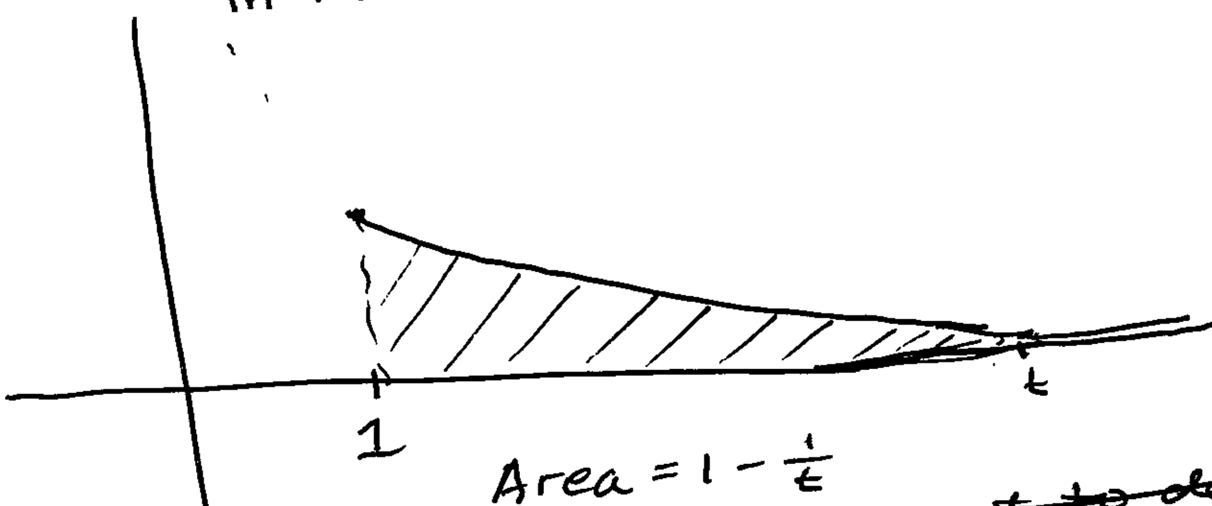
Example (Explanation)

Calculate $\int_1^t \frac{1}{x^2}\,dx$

$= \int_1^t x^{-2}\,dx = \frac{x^{-1}}{-1}\Big|_1^t = \left[-\frac{1}{x}\right]_1^t$

$= \left(\frac{-1}{t}\right) - \left(\frac{-1}{1}\right) = \frac{-1}{t} + 1 = 1 - \frac{1}{t}.$

In terms of Area



Area $= 1 - \frac{1}{t}$

~~We Suppose we want to define~~

We define the area under the curve for $x \geq 1$ as

$\lim_{t \to \infty} \left(1 - \frac{1}{t}\right) = 1$

$\downarrow$
0

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We define the improper integral (Type I)

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx$$

Provided the limit exists.

In general we define the improper integral as

① $$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided the limit exists.

② $$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

③ $$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

For any real number a.

If the limit exists, then we say that the integral is convergent. If the limit does not exist, then we say that the ~~the limit is~~ integral is divergent.

②

Example: Determine whether the integral $\int_1^{\infty} \frac{1}{x}\,dx$ is convergent or divergent.

SOLUTION

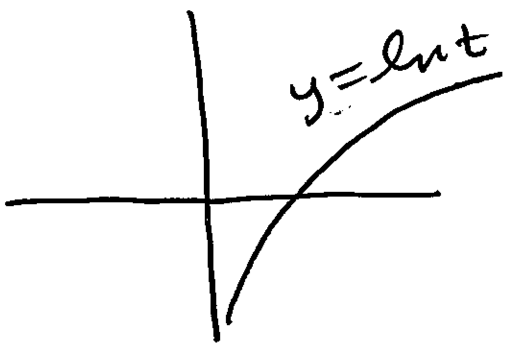
$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t \to \infty} \int_1^{t} \frac{1}{x}\,dx$$

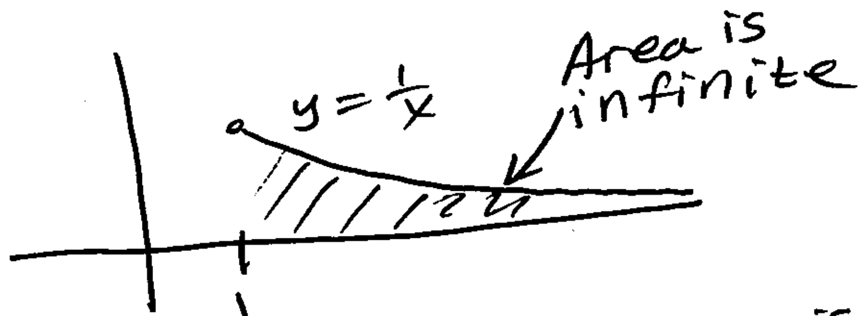
$$= \lim_{t \to \infty} \Big[\ln|x|\Big]_1^t$$

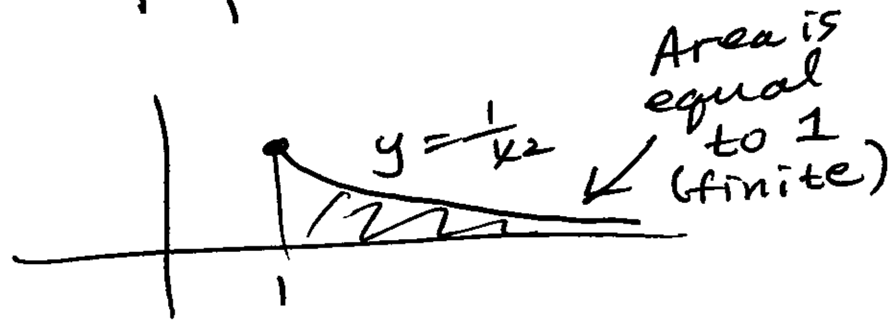
$$= \lim_{t \to \infty} \ln(t) - \overbrace{\ln(1)}^{=0}$$

$$= \infty$$

divergent.







Example $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

Determine whether the integral is convergent or divergent. If it is convergent, then calculate the limit.

SOLUTION

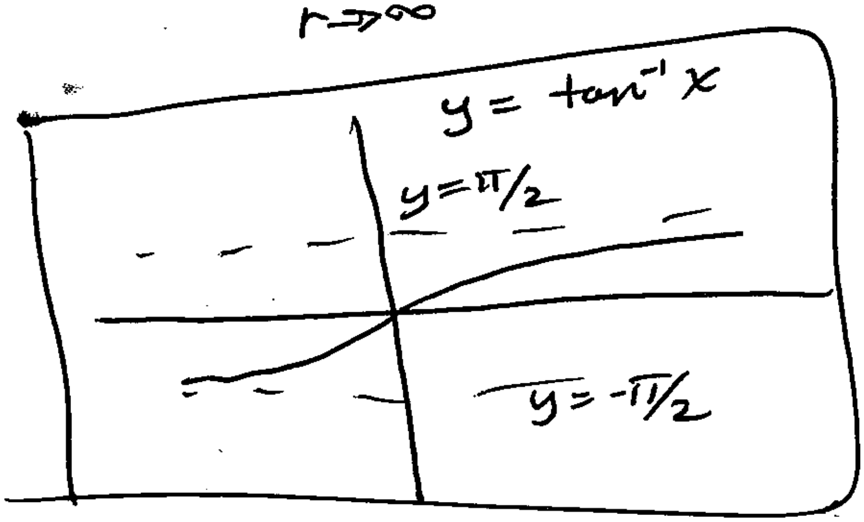
$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{1+x^2}\,dx + \lim_{r\to\infty} \int_{0}^{r} \frac{1}{1+x^2}\,dx$$

$$= \lim_{t\to-\infty} \Big[\tan^{-1}(x)\Big]_t^0 + \lim_{r\to\infty} \Big[\tan^{-1}(x)\Big]_0^r$$

$$= \lim_{t\to-\infty} \big(\underbrace{\tan^{-1}(0)}_{=0} - \tan^{-1}(t)\big)$$

$$+ \lim_{r\to\infty} \big[\tan^{-1}(r) - \underbrace{\tan^{-1}(0)}_{=0}\big]$$



$$= \lim_{t\to-\infty} \big(-\tan^{-1}(t)\big)$$

$$+ \lim_{r\to\infty} \tan^{-1} r$$

$$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2}$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Convergent.

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Example: For which values of p is the integral

$$\int_1^\infty \frac{1}{x^p}\,dx \text{ convergent?}$$

SOLUTION

CASE I $p=1$

$\int_1^\infty \frac{1}{x}\,dx$ which is divergent from the previous example.

CASE: $p \neq 1$ $\quad \int_1^\infty \frac{1}{x^p}\,dx = \int_1^\infty x^{-p}\,dx$

$$= \lim_{t\to\infty} \int_1^t x^{-p}\,dx$$

$$= \lim_{t\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^t$$

$$= \cancel{\lim_{t\to\infty} x^{-p+1}} \lim_{t\to\infty}\left[\frac{t^{-p+1}}{1-p}\right] - \left[\underbrace{\frac{1^{-p+1}}{-p+1}}_{\frac{1}{1-p}}\right]$$

$$= \frac{1}{1-p}\left(\lim_{t\to\infty} t^{-p+1}\right) - \frac{1}{1-p}$$

- If $p > 1$

$$\lim_{t\to\infty} t^{-p+1} = \lim_{t\to\infty} t^{-(p-1)} = \lim_{t\to\infty} \frac{1}{t^{(p-1)}}$$

note: $p-1>0$ becaus $p>1$

~~In general~~ $\cancel{\lim_{t\to\infty} \frac{1}{t^1}}$

In general $\lim_{t\to\infty} \frac{1}{t^r} = 0$ if $r > 0$.

Thereforefore $\lim_{t\to\infty} \frac{1}{t^{p-1}} = 0$ if $p > 1$

$\boxed{\text{So convergent if } p > 1}$

- If $p < 1$

$$\lim_{t\to\infty} t^{-p+1} = \lim_{t\to\infty} t^{(1-p)}$$

Note: $p < 1$
$\Rightarrow 0 < 1 - p$

So $\lim_{t\to\infty} t^{(1-p)} = \infty$ $\boxed{\text{Divergent if } p < 1}$

Answer: $\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and divergent if $p \leq 1$.

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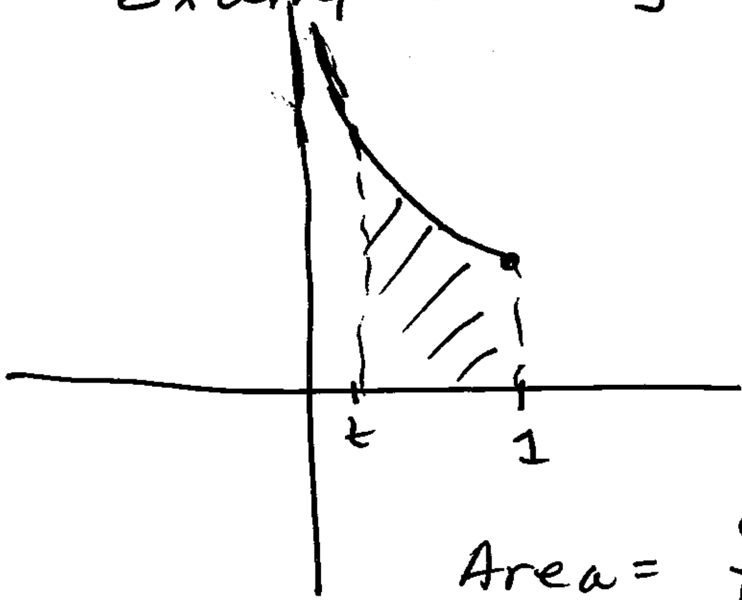
Type 2 Discontinuous Integrands.

Example: $y = \frac{1}{x}$, $t \le x \le 1$
$0 < t$



$$\text{Area} = \int_t^1 \frac{1}{x}\,dx = \Big[\ln|x|\Big]_t^1$$

$$= \ln(1) - \ln(t)$$
$$= 0 - \ln(t) = -\ln t$$

Note that $y = \frac{1}{x}$ is continuous on $[t, 1]$ if $0 < t < 1$.

We define the improper integral (type 2) as

$$\int_0^1 \frac{1}{x}\,dx = \lim_{t \to 0^+} \int_t^1 \frac{1}{x}\,dx$$

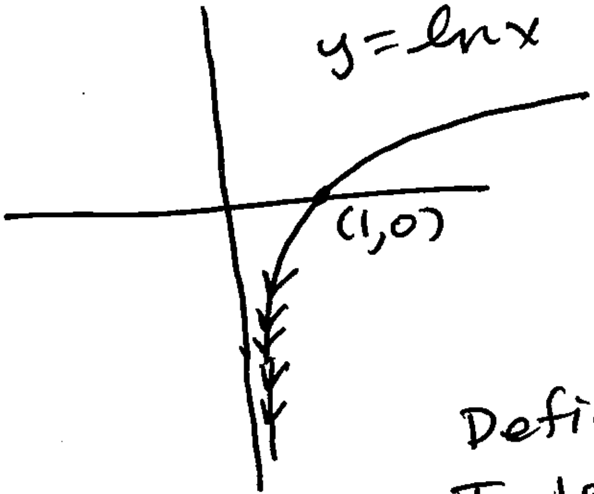
<u>Note</u>: $\frac{1}{x}$ is undefined at $x = 0$

we get

$$\int_0^1 \frac{1}{x}\,dx = \lim_{t \to 0^+} \int_t^1 \frac{1}{x}\,dx = \lim_{t \to 0^+} \ln|x|\Big]_t^1$$

$$= \lim_{t \to 0^+} (\underbrace{\ln 1}_{0} - \ln t) = \lim_{t \to 0^+} (-\underbrace{\ln t}_{\to -\infty}) = \infty \quad (1)$$

Divergent

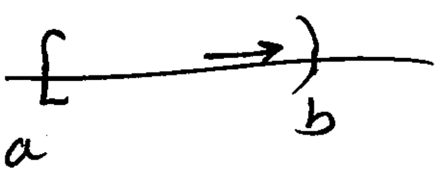


Definition of Improper Integral (Type 2)

① If f is ~~dis~~ continuous on $[a, b)$ but discontinuous at b,

$a \le x < b$

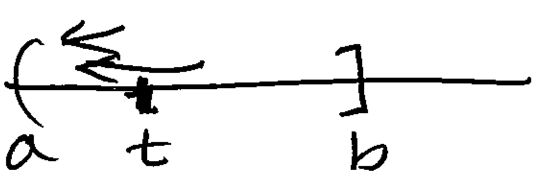
then $\displaystyle\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$



② If f is discontinuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \cancel{\lim_{t \to a^+} f(x)\,dx}$$

$$= \lim_{t \to a^+} \int_t^b f(x)\,dx$$



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③ If f is discontinuous at c where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, ~~then~~ then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

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