Math 185 Weds 3/10

§7.8 #13 $\int_{-\infty}^{\infty} xe^{-x^2}dx = \underbrace{\int_{-\infty}^{0} xe^{-x^2}dx}_{\text{I}} + \underbrace{\int_{0}^{\infty} xe^{-x^2}dx}_{\text{II}}$

Indefinite Integral

$\int xe^{-x^2}dx = \frac{-1}{2}\int e^u du = \frac{-1}{2}e^u = \frac{-1}{2}e^{-x^2}$

$u = -x^2$

$du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

I $\int_{-\infty}^{0} xe^{-x^2}dx = \lim_{t\to-\infty} \int_t^0 xe^{-x^2}dx$

$= \lim_{t\to-\infty} \left[-\frac{1}{2}e^{-x^2}\right]_t^0$

$= \lim_{t\to-\infty} \left[\frac{-1}{2}e^0 - \frac{-1}{2}e^{-t^2}\right]$

$= -\frac{1}{2}$

$\boxed{\lim_{t\to-\infty} e^{-t^2} = 0}$

II $\int_0^{\infty} xe^{-x^2}dx = \lim_{t\to\infty} \int_0^t xe^{-x^2}dx$

$= \lim_{t\to\infty} \left[-\frac{1}{2}e^{-x^2}\right]_0^t = \lim_{t\to\infty} \left[\frac{-1}{2}e^{-t^2} - \frac{-1}{2}e^0\right]$

$= \frac{1}{2}$

Answer: $-\frac{1}{2} + \frac{1}{2} = 0$

§7.8 #19 $\int_0^\infty s e^{-5s}\,ds = \lim_{t\to\infty} \int_0^t s e^{-5s}\,ds$

Indefinite Integral

$\int s e^{-5s}\,ds =$

Integrate by Parts

$u = s \qquad dv = e^{-5s}\,ds$

$du = ds \qquad v = -\frac{1}{5}e^{-5s}$

$$\int u\,dv = uv - \int v\,du$$

$$= s\left(-\frac{1}{5}e^{-5s}\right) - \int -\frac{1}{5}e^{-5s}\,ds$$

$$= -\frac{se^{-5s}}{5} + \frac{1}{5}\left(-\frac{1}{5}e^{-5s}\right)$$

$$= -\frac{s}{5}e^{-5s} - \frac{1}{25}e^{-5s}$$

$$= \lim_{t\to\infty}\left[-\frac{s}{5}e^{-5s} - \frac{1}{25}e^{-5s}\right]_0^t$$

$$= \lim_{t\to\infty}\left[\overset{(A)}{-\frac{t}{5}e^{-5t}} \overset{(B)}{-\frac{1}{25}e^{-5t}}\right] - \left[-\frac{0}{5}e^{-5(0)} - \frac{1}{25}e^{-5(0)}\right]$$

$$-\frac{1}{25}e^{-5(0)} = -\frac{1}{25}(1)$$

(A) $\lim_{t\to\infty} -\frac{t}{5}e^{-5t} = -\frac{1}{5}\lim_{t\to\infty}\frac{t}{e^{5t}}$ $\frac{\infty}{\infty}$ form

Use L'Hospital's Rule

$$\overset{L'H}{=} \frac{-1}{5}\lim_{t\to\infty}\frac{1}{5e^{5t}} = 0$$

(B) $\lim_{t\to\infty} -\frac{1}{25}e^{-5t} = 0$

Answer: $0 - \left(-\frac{1}{25}\right) = \frac{1}{25}$

§7.8 #39 $\displaystyle\int_0^2 z^2 \ln z \, dz$

$z^2 \ln z$ is discont. at $z=0$.

$$= \lim_{t \to 0^+} \int_t^2 z^2 \ln z \, dz$$

$$\int z^2 \ln z \, dz$$

$u = \ln z \qquad dv = z^2 \, dz$

$du = \frac{1}{z} dz \qquad v = \frac{z^3}{3}$

$$\int u\,dv = uv - \int v\,du$$

$$= \frac{z^3}{3} \ln z - \int \frac{z^3}{3} \cdot \frac{1}{z} dz$$

$$= \frac{z^3}{3} \ln z - \int \frac{z^2}{3} dz$$

$$= \frac{z^3}{3} \ln z - \frac{1}{3}\frac{z^3}{3} = \frac{z^3}{3} \ln z - \frac{1}{9} z^3$$

$$= \lim_{t \to 0^+} \left[\frac{z^3}{3} \ln z - \frac{z^3}{9} \right]_t^2 = \left(\frac{2^3}{3} \ln 2 - \frac{2^3}{9} \right) - \lim_{t \to 0^+} \left[\underbrace{\frac{t^3}{3} \ln t}_{\text{I}} - \underbrace{\frac{t^3}{9}}_{\text{II}} \right]$$

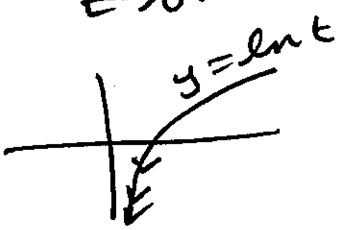
II $\displaystyle\lim_{t \to 0^+} \frac{t^3}{9} = 0$

I $\displaystyle\lim_{t \to 0^+} \overbrace{\frac{t^3}{3}}^{0} \overbrace{\ln t}^{-\infty} \quad 0 \cdot -\infty = \frac{1}{3} \lim_{t \to 0^+} \frac{\overbrace{\ln t}^{-\infty}}{t^{-3}} \overset{L'H}{=} \frac{1}{3} \lim_{t \to 0} \frac{1/t}{-3t^{-4}}$

$\frac{1}{t^3} \to \infty$

$$= -\frac{1}{9} \lim_{t \to 0} \frac{1}{t} \cdot t^4$$

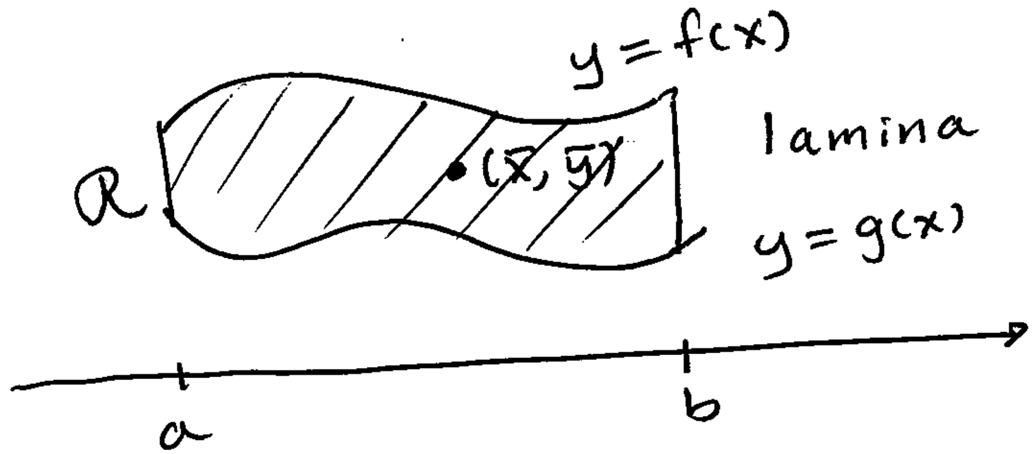
$$= -\frac{1}{9} \lim_{t \to 0} t^3 = 0$$



Answer: $\displaystyle\frac{3 \cdot 2^3}{3 \cdot 3} \ln 2 - \frac{2^3}{9} = \frac{8}{9} \left(3 \ln(2) - 1 \right)$

Quiz Thurs
7.8, 8.1, 8.2

§ 8.3 Notes continued



Centroid $(\bar{x}, \bar{y})$ is given by

$$\bar{x} = \frac{1}{A} \int_a^b x\,[f(x) - g(x)]\,dx$$

$$\bar{y} = \frac{1}{A} \int_a^x \frac{1}{2}\left\{[f(x)]^2 - [g(x)]^2\right\} dx$$

§ 9.3 Separable Equations

HW §9.3 #1-18

A <u>differential equation</u> is an equation that contains an unknown function and one or more of its derivatives.

Here are some examples.

$$y' = xy$$

$$y'' + 2y' + y = 0$$

$$\frac{d^3y}{dx^3} + x\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = e^{-x}$$

In each of these equations, y is an unknown function of x.

The <u>order</u> of a differential equation is the ~~order~~ degree of the highest derivative in the equation.

A function f is called a <u>solution</u> of a differential equation if the equation is satisfied when $y=f(x)$ and its derivatives are substituted into the equation.

Separable Equations

A separable equation is a first order DE that can be written in the form

$$\frac{dy}{dx} = g(x)\,f(y)$$

If $f(y) \neq 0$, we can write

$$\frac{dy}{dx} = \frac{g(x)}{\left(1/f(y)\right)} = \frac{g(x)}{h(y)}$$

where $h(y) = \frac{1}{f(y)}$

we have

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

So solve we move this around to get $h(y)\,dy = g(x)\,dx$

and integrate

$$\int h(y)\,dy = \int g(x)\,dx$$

To justify this, we differentiate both sides

$$\frac{d}{dx}\left(\int h(y)\,dy\right) = \frac{d}{dx}\left(\int g(x)\,dx\right)$$

$$\frac{d}{dy}\left(\int h(y)\,dy\right)\cdot\frac{dy}{dx} = \frac{d}{dx}\left(\int g(x)\,dx\right)$$

$$h(y)\cdot\frac{dy}{dx} = g(x)$$

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

EXAMPLE $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}$

SOLUTION

$$y^2\,dy = x^2\,dx$$

$$\int y^2\,dy = \int x^2\,dx$$

$$\frac{y^3}{3} = \frac{x^3}{3} + C$$

$$y^3 = x^3 + 3C \qquad K = 3C$$

$$y = \sqrt[3]{x^3 + K} \quad \square$$

We sometimes want to find a solution that also satisfies an initial condition $y(x_0) = y_0$. This is called an initial value problem.

Example: Solve the initial value problem

$$\frac{dy}{dx} = \frac{y\cos x}{1+y^2}, \quad y(0)=1$$

when $x=0$, $y=1$

SOLUTION

$$\frac{dy}{dx} = \left(\frac{y}{1+y^2}\right)\cos x$$

$$\int \left(\frac{1+y^2}{y}\right) dy = \int \cos x \, dx$$

$$\int \Big(\overset{\text{A}}{\frac{1}{y} + \frac{y^2}{y}}\Big) dy = \overset{\text{B}}{\int \cos x \, dx}$$

A $\int \left(\frac{1}{y} + y\right) dy = \ln|y| + \frac{y^2}{2}$

B $\int \cos x \, dx = \sin x$

$$\ln|y| + \frac{y^2}{2} = \sin x + C$$

We solve for C using the initial condition $x=0$, $y=1$

$$\ln|1| + \frac{(1)^2}{2} = \sin(0) + C$$

$$0 + \frac{1}{2} = 0 + C$$

$$C = \frac{1}{2}$$

$$\boxed{\ln|y| + \frac{y^2}{2} = \sin x + \frac{1}{2}}$$